



Lossy Transmission Lines Terminated by Parallel Connected RC-Loads in Series Connected to L-Load (II)

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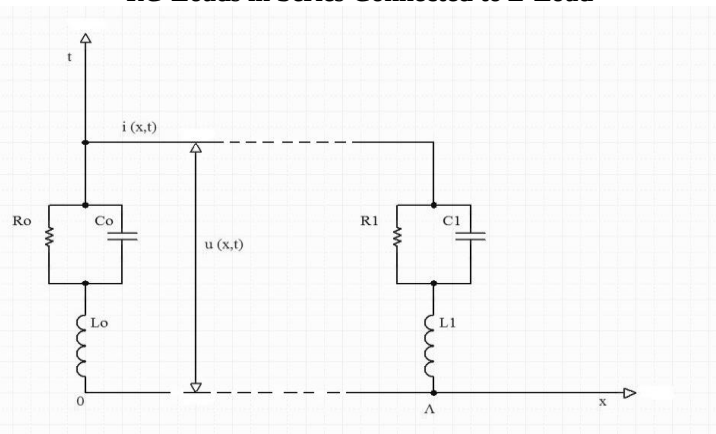
Abstract: The present paper is a second part of the previous one (Angelov and Zahariev, 2013) where a lossy transmission line terminated by the same circuit of nonlinear loads is considered. Here we introduce an operator acting in a suitable function space whose fixed point is an oscillatory solution of the neutral system obtained in the first paper cited above. Then we formulate conditions for the existence-uniqueness of an oscillatory solution. The conditions obtained are easily verifiable inequalities. We demonstrate the advantages of our method by a numerical example.

Keywords: lossy transmission line, mixed problem for hyperbolic system, neutral equation, oscillatory solution, fixed point theorem, Kirchhoff's law.

I. Introduction

The general theory and various applications of transmission line theory can be found in many monographs ([9]-[17]). In previous papers ([2]-[7]) we have made an analysis of transmission lines (both lossless and lossy ones) loaded by various configurations of nonlinear elements. Motivations for the present considerations in the first part of the paper [8] are given. Since we continue the investigations from [8] this inflicts to recall briefly the main results obtained.

Figure 1 Lossless Transmission Line Terminated by Parallel Connected RC-Loads in Series Connected to L-Load



As a consequence of Kirchhoff's law the boundary conditions are derived (cf. Figure 1) and then the mixed problem for the hyperbolic transmission line system is formulated. The reducing of the mixed problem for the hyperbolic system leads to an initial value problem for neutral equations on the boundary. We also show that natural solutions are oscillatory ones, tending to zero at infinity. The primary goal of the present paper is to introduce a suitable operator whose fixed point is an oscillatory solution of the neutral system. By fixed point method given in [1] we obtain an existence-uniqueness of an oscillatory solution and demonstrate the application of our method to specific problems. The paper consists of five sections. In Section II we recall the

formulation of the mixed problem for the hyperbolic system (Telegrapher equations) and give some estimates of the characteristics of the nonlinear loads. In Section III we introduce an operator presentation of the oscillatory problem. The key role plays Lemma 3.3. In Section IV we establish existence-uniqueness of oscillatory solution tending to zero at infinity. In Section V we demonstrate the advantages of our method on a numerical example. We point out that for applications it is necessary to verify just simple inequalities.

II. The Mixed Problem and Neutral System

In the first part (see [8]) we have derived boundary conditions of the mixed problem for lossy transmission line system and we have formulated the mixed problem: to find functions $u(x,t)$ and $i(x,t)$, satisfying the system

$$\begin{aligned} C \frac{\partial u(x,t)}{\partial t} + \frac{\partial i(x,t)}{\partial x} + Gu(x,t) &= 0, \\ L \frac{\partial i(x,t)}{\partial t} + \frac{\partial u(x,t)}{\partial x} + Ri(x,t) &= 0 \end{aligned} \quad (2.1)$$

for $(x,t) \in \Pi = \{(x,t) \in \Pi^2 : (x,t) \in [0,\Lambda] \times [0,\infty)\}$ and such that the following initial

$$u(x,0) = u_0(x), \quad i(x,0) = i_0(x), \quad x \in [0,\Lambda] \quad (2.2)$$

and boundary conditions

$$\begin{aligned} \left[u_{G_0 C_0} \frac{dC_0(u_{G_0 C_0})}{du_{G_0 C_0}} + C_0(u_{G_0 C_0}) \right] \frac{du_{G_0 C_0}}{dt} &= -i(0,t) - G_0(u_{G_0 C_0}), \\ \left[i(0,t) \frac{dL_0(i(0,t))}{di_{G_0 C_0}} + L_0(i(0,t)) \right] \frac{di(0,t)}{dt} &= u(0,t) - u_{G_0 C_0}(t) + E_0(t), \end{aligned} \quad (2.3)$$

$$\begin{aligned} \left[u_{G_1 C_1} \frac{dC_1(u_{G_1 C_1})}{du_{G_1 C_1}} + C_1(u_{G_1 C_1}) \right] \frac{du_{G_1 C_1}}{dt} &= i(\Lambda,t) - G_1(u_{G_1 C_1}), \\ \left[i(\Lambda,t) \frac{dL_1(i(\Lambda,t))}{di_{G_1 C_1}} + L_1(i(\Lambda,t)) \right] \frac{di(\Lambda,t)}{dt} &= -u(\Lambda,t) + u_{G_1 C_1}(t) - E_1(t) \end{aligned} \quad (2.4)$$

are satisfied. Here L, C, R and G are prescribed specific parameters of the line and $\Lambda > 0$ is its length.

Let us recall that we deal with nonlinear elements whose characteristics are:

$$R_p(i) = \sum_{n=1}^m r_n^{(p)} i^n, \quad (p=0,1), \quad L_p(i) = \sum_{n=0}^m l_n^{(p)} i^n, \quad \tilde{L}_p(i) = i \cdot L_p(i) = i \cdot \sum_{n=0}^m l_n^{(p)} i^n, \quad \tilde{C}_p(u) = u C_p(u)$$

such that

$$\begin{aligned} \frac{d\tilde{L}_p(i)}{dt} &= \frac{d(i L_p(i))}{dt} = \frac{di}{dt} \left(L_p(i) + i \frac{dL_p(i)}{di} \right), \\ \frac{d\tilde{C}_p(u)}{dt} &= \frac{d(u C_p(u))}{dt} = \frac{du}{dt} \left(C_p(u) + u \frac{dC_p(u)}{du} \right), \quad (p=0,1). \end{aligned}$$

From Section IV of the first part (see [8]) we have

$$\begin{aligned} C_p(u) &= \frac{c_p \sqrt[h]{\Phi_p}}{\sqrt[h]{\Phi_p} - u}; \quad \tilde{C}_p(u) = u C_p(u) = \frac{c_p \sqrt[h]{\Phi_p} u}{\sqrt[h]{\Phi_p} - u}; \quad c_p > 0; \quad \Phi_p > 0; \quad h \in [2,3]; \quad |u| \leq \phi_0 < \min\{\Phi_0, \Phi_1\}; \\ \frac{dC_p(u)}{du} &= \frac{c_p \sqrt[h]{\Phi_p}}{h} (\Phi_p - u)^{-\frac{1+h}{h}}; \quad \frac{d^2 C_p(u)}{du^2} = \frac{c_p \sqrt[h]{\Phi_p}}{h} \frac{1+h}{h} (\Phi_p - u)^{-\frac{1+2h}{h}}; \\ \frac{d\tilde{C}_p(u)}{du} &= c_p \sqrt[h]{\Phi_p} \left(\Phi_p - \frac{h-1}{h} u \right) (\Phi_p - u)^{-\frac{1}{h}}; \quad \frac{d^2 \tilde{C}_p(u)}{du^2} = \frac{c_p \sqrt[h]{\Phi_p}}{h^2} \frac{-2h^2 \Phi_p + (2h^2 + h + 1)u}{(\Phi_p - u)^{2+\frac{1}{h}}}. \end{aligned}$$

Since $|u| \leq \phi_0 < \min\{\Phi_0, \Phi_1\} < \min\left\{\frac{h}{h-1}\Phi_0, \frac{h}{h-1}\Phi_1\right\}$ the derivative $\frac{d\tilde{C}_p(u)}{du} > 0$. Further on we have

$$\left| \frac{d\tilde{C}_p(u)}{du} \right| \leq \frac{2c_p \sqrt[h]{\Phi_p} (h\Phi_p + \phi_0)}{h^2 \sqrt[h]{(\Phi_p - \phi_0)^{1+2h}}} \equiv \tilde{C}_p^{(1)}, \quad |u| \leq \phi_0 < \min\left\{\frac{h}{h+1}, \Phi_0, \Phi_1\right\} \Rightarrow$$

$$\frac{d\tilde{C}_p(u)}{du} \geq \frac{c_p \sqrt[4]{\Phi_p}}{(\Phi_p + \phi_0)^{\frac{1+h}{h}}} \left(\Phi_p - \frac{h-1}{h} \phi_0 \right) \equiv \hat{C}_p^{(1)}, \left| \frac{d^2 \tilde{C}_p(u)}{du^2} \right| \leq \frac{c_p \sqrt[4]{\Phi_p}}{h^2} \frac{2h^2 \Phi_p + (2h^2 + h + 1) \phi_0}{(\Phi_p - \phi_0)^{2+\frac{1}{h}}} \equiv \tilde{C}_p^{(2)}.$$

Since $\tilde{L}_p(i) = i \cdot L_p(i) = i \cdot \sum_{n=0}^m l_n^{(p)} i^n$ we have

$$\frac{d\tilde{L}_p(i)}{di} = i \frac{dL_p(i)}{di} + L_p(i) = i \sum_{n=1}^m n l_n^{(p)} i^{n-1} + \sum_{n=1}^m l_n^{(p)} i^n = \sum_{n=1}^m (n+1) l_n^{(p)} i^n.$$

Assumptions(C): $\frac{e^{\mu T_0}(W_0 + J_0)}{2} \leq \phi_0$; $\frac{e^{\mu T_0}(W_0 + J_0)}{2Z_0} \leq I_0$.

It follows $|u(0, t)|, |u(\Lambda, t)|, |i(0, t)|, |i(\Lambda, t)| \leq \phi_0$.

Assumptions (L): $|i| \leq I_0 < \infty \Rightarrow \frac{d\tilde{L}_p(i)}{di} = \sum_{n=1}^m (n+1) l_n^{(p)} i^n \geq \hat{L}_p^{(1)} > 0$,

$$\left| \frac{d\tilde{L}_p(i)}{di} \right| \leq \sum_{n=1}^m (n+1) l_n^{(p)} I_0^n = \tilde{L}_p^{(1)}, \left| \frac{d^2 \tilde{L}_p(i)}{di^2} \right| \leq \sum_{n=1}^{m-1} n(n+1) l_n^{(p)} I_0^{n-1} = \tilde{L}_p^{(2)}.$$

Assumptions (G): $G_0(i_{R_0 L_0}) = \sum_{n=1}^m g_n^{(0)} (i_{R_0 L_0})^n$, $G_1(i_{R_1 L_1}) = \sum_{n=1}^m g_n^{(1)} (i_{R_1 L_1})^n$.

By the transformation

$$\begin{cases} u(x, t) = \frac{1}{2\sqrt{C}} e^{-\frac{R}{L}t} W(x, t) - \frac{1}{2\sqrt{C}} e^{-\frac{R}{L}t} J(x, t), \\ i(x, t) = \frac{1}{2\sqrt{L}} e^{-\frac{R}{L}t} W(x, t) + \frac{1}{2\sqrt{L}} e^{-\frac{R}{L}t} J(x, t) \end{cases}$$

under Heaviside condition $R/L = G/C$ system (2.1) becomes

$$\frac{\partial W(x, t)}{\partial t} + \frac{1}{\sqrt{LC}} \frac{\partial W(x, t)}{\partial x} = 0, \quad \frac{\partial J(x, t)}{\partial t} - \frac{1}{\sqrt{LC}} \frac{\partial J(x, t)}{\partial x} = 0$$

with initial conditions

$$W(x, 0) = \sqrt{C} u(x, 0) + \sqrt{L} i(x, 0) = \sqrt{C} u_0(x) + \sqrt{L} i_0(x) \equiv W_0(x),$$

$$J(x, 0) = -\sqrt{C} u(x, 0) + \sqrt{L} i(x, 0) = -\sqrt{C} u_0(x) + \sqrt{L} i_0(x) \equiv J_0(x), \quad x \in [0, \Lambda]$$

We assume that the unknown functions are $W(0, t) = W(t)$, $J(t) = J(\Lambda, t)$. Then in view of

$$W(0, t) = W(\Lambda, t + T), \quad J(0, t + T) = J(\Lambda, t) \Rightarrow W(0, t - T) = W(\Lambda, t), \quad J(0, t) = J(\Lambda, t - T)$$

the boundary conditions yield a neutral system:

$$\begin{aligned} \frac{du_{G_0 C_0}(t)}{dt} &= \frac{e^{-\frac{R}{L}t} W(t) + e^{-\frac{R}{L}(t-T)} J(t-T) - 2\sqrt{L} G_0(u_{G_0 C_0}(t))}{2\sqrt{L} d\tilde{C}_0(u_{G_0 C_0})/du_{G_0 C_0}}; \\ \frac{d}{dt} \left(e^{-\frac{R}{L}t} W(t) + e^{-\frac{R}{L}(t-T)} J(t-T) \right) &= \frac{Z_0 e^{-\frac{R}{L}t} W(t) - Z_0 e^{-\frac{R}{L}(t-T)} J(t-T) - 2\sqrt{L} u_{G_0 C_0}(t) + 2\sqrt{L} E_0(t)}{d\tilde{L}_0(i(0, t))/di_{G_0 C_0}}; \\ \frac{du_{G_1 C_1}(t)}{dt} &= \frac{e^{-\frac{R}{L}(t-T)} W(t-T) + e^{-\frac{R}{L}t} J(t) - 2\sqrt{L} G_1(u_{G_1 C_1}(t))}{2\sqrt{L} dC_1(u_{G_1 C_1})/du_{G_1 C_1}}; \\ \frac{d}{dt} \left(e^{-\frac{R}{L}(t-T)} W(t-T) + e^{-\frac{R}{L}t} J(t) \right) &= \frac{-Z_0 e^{-\frac{R}{L}(t-T)} W(t-T) + Z_0 e^{-\frac{R}{L}t} J(t) + 2\sqrt{L} u_{G_1 C_1}(t) - 2\sqrt{L} E_1(t)}{d\tilde{L}_1(i(\Lambda, t))/di_{G_1 C_1}}. \end{aligned} \quad (2.5)$$

III. Operator Presentation of the Oscillatory

We notice that if the neutral system (2.5) has a periodic solution

$(u_{G_0C_0}(t), W(t), u_{G_1C_1}(t), J(t))$ then functions $e^{-\frac{R}{L}t}W(t), e^{-\frac{R}{L}t}J(t)$ are oscillatory ones and vanishing exponentially at infinity. Therefore we put $\widehat{W}(t) = e^{-\frac{R}{L}t}W(t), \widehat{J}(t) = e^{-\frac{R}{L}t}J(t)$ and then we can state the problem for existence-uniqueness of an oscillatory solution vanishing at infinity of the following system (we denote by $\widehat{W}(t), \widehat{J}(t)$ again by $W(t), J(t)$):

$$\begin{aligned} \frac{du_{G_0C_0}(t)}{dt} &= \frac{W(t) + J(t-T) - 2\sqrt{L}G_0(u_{G_0C_0}(t))}{2\sqrt{L} \widetilde{dC_0}(u_{G_0C_0})/du_{G_0C_0}}, t \in [T; \infty); \\ \frac{dW(t)}{dt} &= -\frac{dJ(t-T)}{dt} + \frac{Z_0W(t) - Z_0J(t-T) - 2\sqrt{L}u_{G_0C_0}(t) + 2\sqrt{L}E_0(t)}{\widetilde{dL_0}(i(0,t))/di_{G_0C_0}}, t \in [T; \infty); \\ \frac{du_{G_1C_1}(t)}{dt} &= \frac{W(t-T) + J(t) - 2\sqrt{L}G_1(u_{G_1C_1}(t))}{2\sqrt{L} \widetilde{dC_1}(u_{G_1C_1})/du_{G_1C_1}}, t \in [T; \infty); \\ \frac{dJ(t)}{dt} &= -\frac{dW(t-T)}{dt} + \frac{-Z_0W(t-T) + Z_0J(t) + 2\sqrt{L}u_{G_1C_1}(t) - 2\sqrt{L}E_1(t)}{\widetilde{dL_1}(i(\Lambda,t))/di_{G_1C_1}}, t \in [T; \infty) \\ u_{G_0C_0}(T) &= u_{G_0C_0}^T, u_{G_1C_1}(T) = u_{G_1C_1}^T, W(t) = W_0(t), J(t) = J_0(t), \\ \frac{dW(t)}{dt} &= \frac{dW_0(t)}{dt}, \frac{dJ(t)}{dt} = \frac{dJ_0(t)}{dt}, t \in [0, T]. \end{aligned} \quad (3.1)$$

We introduce an operator presentation of the oscillatory problem and by a fixed point theorem in uniform spaces given in [1] we establish an existence-uniqueness of oscillatory solution.

Now we are able to formulate the main problem: to find a solution of (3.1) with advanced prescribed zeros on $[t_0, \infty)$, $T \equiv t_0$, where $\widetilde{W}_0(t), \widetilde{J}_0(t)$ are prescribed initial oscillating functions on the interval $[0, t_0]$.

Let $S_T = \{\tau_k\}_{k=0}^n, n \in \mathbb{N}$ be the set of zeros of the initial function, that is, $\widetilde{W}_0(\tau_k) = 0, \widetilde{J}_0(\tau_k) = 0$ such that $\tau_0 = 0, \tau_n = T \equiv t_0$. Besides $\max\{\tau_{k+1} - \tau_k : k = 0, 1, \dots, n\} \leq T_0$.

Let $S = \{t_k\}_{k=0}^\infty$ be a strictly increasing sequence of real numbers satisfying the following conditions (C):

(C1) $\lim_{k \rightarrow \infty} t_k = \infty$;

(C2) for every k there is $s < k$ such that $t_k - T = t_s$ where $t_s \in S_T \cup S$.

It follows, that

(C3) $0 < \inf\{t_{k+1} - t_k : k = 0, 1, 2, \dots\} \leq \sup\{t_{k+1} - t_k : k = 0, 1, 2, \dots\} = T_0 < \infty$.

Introduce the set $C^1[t_0, \infty)$ consisting of all oscillatory continuous and bounded functions differentiable with bounded derivatives on every interval $[t_k, t_{k+1}]$.

Remark 3.1 Let us note that the left and the right derivative at the point t_k of both functions $W(\cdot), J(\cdot) \in C^1[t_0, \infty)$ could not be coincided. That is why we introduce a suitable topology for continuous functions with piecewise continuous derivatives.

Introduce the sets

$$\begin{aligned} M_0 &= \left\{ u_{G_0C_0}(\cdot) \in C^1[t_0, \infty) : u_{G_0C_0}(t_k) = 0 \wedge |u_{G_0C_0}(t)| \leq U_{G_0} e^{-\frac{R}{L}t}, t \in [t_k, t_{k+1}]; u_{G_0C_0}(t_0) = 0 \right\}, \\ M_W &= \left\{ W(\cdot) \in C^1[t_0, \infty) : W(t_k) = 0 \wedge |W(t)| \leq W_0 e^{-\frac{R}{L}t}, t \in [t_k, t_{k+1}] \right\}, \\ M_1 &= \left\{ u_{G_1C_1}(\cdot) \in C^1[t_0, \infty) : u_{G_1C_1}(t_k) = 0 \wedge |u_{G_1C_1}(t)| \leq U_{G_1} e^{-\frac{R}{L}t}, t \in [t_k, t_{k+1}]; u_{G_1C_1}(t_0) = 0 \right\}, \\ M_J &= \left\{ J(\cdot) \in C^1[t_0, \infty) : J(t_k) = 0 \wedge |J(t)| \leq J_0 e^{-\frac{R}{L}t}, t \in [t_k, t_{k+1}] \right\}. \end{aligned}$$

The following assumptions will be hold:

Assumption (IN): $|\tilde{W}_0(t)| \leq W_0 e^{-\beta} e^{-\frac{R}{L}(t+T)}$; $|\tilde{J}_0(t)| \leq J_0 e^{-\beta} e^{-\frac{R}{L}(t+T)}$, $t \in [0, T]$.

It follows $|W(t-T)| \leq W_0 e^{-\beta} e^{-\frac{R}{L}t}$; $|J(t-T)| \leq J_0 e^{-\beta} e^{-\frac{R}{L}t}$, $t \in [T, 2T]$.

Assumption (II): $\frac{W_0 + J_0}{2\sqrt{C}} \leq \phi_0$, $U_{G_0}, U_{G_1} \leq \phi_0$.

Remark 3.2 It follows that the functions from M_0, M_W, M_1 and M_J satisfy the inequalities

$$|u_{G_0 C_0}(t)| \leq U_{G_0} e^{\mu(t-t_k)}, |W(t)| \leq W_0 e^{\mu(t-t_k)}, |u_{G_1 C_1}(t)| \leq U_{G_1} e^{\mu(t-t_k)}, |J(t)| \leq J_0 e^{\mu(t-t_k)}, t \in [t_k, t_{k+1}], (k=0,1,2,\dots)$$

where $U_{G_0}, W_0, U_{G_1}, J_0, \mu, \mu T_0 = \mu_0 = \text{const.}$ are positive constants.

Introduce the following family of pseudo-metrics:

$$\begin{aligned} \rho^{(k)}(u_{G_p C_p}, \bar{u}_{G_p C_p}) &= \max \left\{ |u_{G_p C_p}(t) - \bar{u}_{G_p C_p}(t)| : t \in [t_k, t_{k+1}] \right\}, (p=0,1); \\ \rho^{(k)}(W, \bar{W}) &= \max \left\{ |W(t) - \bar{W}(t)| : t \in [t_k, t_{k+1}] \right\}; \rho^{(k)}(J, \bar{J}) = \max \left\{ |J(t) - \bar{J}(t)| : t \in [t_k, t_{k+1}] \right\}; \\ \hat{\rho}^{(k)}(u_{G_p C_p}, \bar{u}_{G_p C_p}) &= \max \left\{ |u_{G_p C_p}(t) - \bar{u}_{G_p C_p}(t)| : t \in [t_0, t_{k+1}] \right\}, (p=0,1); \\ \hat{\rho}^{(k)}(W, \bar{W}) &= \max \left\{ |W(t) - \bar{W}(t)| : t \in [t_0, t_{k+1}] \right\}; \hat{\rho}^{(k)}(J, \bar{J}) = \max \left\{ |J(t) - \bar{J}(t)| : t \in [t_0, t_{k+1}] \right\}; \\ \rho_\mu^{(k)}(u_{G_p C_p}, \bar{u}_{G_p C_p}) &= \max \left\{ e^{-\mu(t-t_k)} |u_{G_p C_p}(t) - \bar{u}_{G_p C_p}(t)| : t \in [t_k, t_{k+1}] \right\}, (p=0,1); \\ \rho_\mu^{(k)}(W, \bar{W}) &= \max \left\{ e^{-\mu(t-t_k)} |W(t) - \bar{W}(t)| : t \in [t_k, t_{k+1}] \right\}; \rho_\mu^{(k)}(J, \bar{J}) = \max \left\{ e^{-\mu(t-t_k)} |J(t) - \bar{J}(t)| : t \in [t_k, t_{k+1}] \right\}; \\ \hat{\rho}_\mu^{(k)}(u_{G_p C_p}, \bar{u}_{G_p C_p}) &= \max \left\{ \rho_\mu^{(0)}(u_{G_p C_p}, \bar{u}_{G_p C_p}), \dots, \rho_\mu^{(k)}(u_{G_p C_p}, \bar{u}_{G_p C_p}) \right\}, (p=0,1); \\ \hat{\rho}_\mu^{(k)}(W, \bar{W}) &= \max \left\{ \rho_\mu^{(0)}(W, \bar{W}), \dots, \rho_\mu^{(k)}(W, \bar{W}) \right\}; \hat{\rho}_\mu^{(k)}(J, \bar{J}) = \max \left\{ \rho_\mu^{(0)}(J, \bar{J}), \dots, \rho_\mu^{(k)}(J, \bar{J}) \right\}; \\ \rho_\mu^{(k)}(\dot{u}_{G_p C_p}, \dot{\bar{u}}_{G_p C_p}) &= \max \left\{ e^{-\mu(t-t_k)} |\dot{u}_{G_p C_p}(t) - \dot{\bar{u}}_{G_p C_p}(t)| : t \in [t_k, t_{k+1}] \right\}, (p=0,1); \\ \rho_\mu^{(k)}(\dot{W}, \dot{\bar{W}}) &= \max \left\{ e^{-\mu(t-t_k)} |\dot{W}(t) - \dot{\bar{W}}(t)| : t \in [t_k, t_{k+1}] \right\}; \rho_\mu^{(k)}(\dot{J}, \dot{\bar{J}}) = \max \left\{ e^{-\mu(t-t_k)} |\dot{J}(t) - \dot{\bar{J}}(t)| : t \in [t_k, t_{k+1}] \right\}; \\ \hat{\rho}_\mu^{(k)}(\dot{u}_{G_p C_p}, \dot{\bar{u}}_{G_p C_p}) &= \max \left\{ \rho_\mu^{(0)}(\dot{u}_{G_p C_p}, \dot{\bar{u}}_{G_p C_p}), \dots, \rho_\mu^{(k)}(\dot{u}_{G_p C_p}, \dot{\bar{u}}_{G_p C_p}) \right\}, (p=0,1); \\ \hat{\rho}_\mu^{(k)}(\dot{W}, \dot{\bar{W}}) &= \max \left\{ \rho_\mu^{(0)}(\dot{W}, \dot{\bar{W}}), \dots, \rho_\mu^{(k)}(\dot{W}, \dot{\bar{W}}) \right\}; \hat{\rho}_\mu^{(k)}(\dot{J}, \dot{\bar{J}}) = \max \left\{ \rho_\mu^{(0)}(\dot{J}, \dot{\bar{J}}), \dots, \rho_\mu^{(k)}(\dot{J}, \dot{\bar{J}}) \right\}. \end{aligned}$$

The following inequalities imply the equivalence of the pseudo-metric families:

$$\begin{aligned} \rho_\mu^{(k)}(u_{G_p C_p}, \bar{u}_{G_p C_p}) &\leq \rho^{(k)}(u_{G_p C_p}, \bar{u}_{G_p C_p}) \leq e^{\mu_0} \rho_\mu^{(k)}(u_{G_p C_p}, \bar{u}_{G_p C_p}), (k=0,1,2,\dots), (p=0,1); \\ \rho_\mu^{(k)}(W, \bar{W}) &\leq \rho^{(k)}(W, \bar{W}) \leq e^{\mu T_0} \rho_\mu^{(k)}(W, \bar{W}), \rho_\mu^{(k)}(J, \bar{J}) \leq \rho^{(k)}(J, \bar{J}) \leq e^{\mu T_0} \rho_\mu^{(k)}(J, \bar{J}), (k=0,1,2,\dots); \end{aligned}$$

It is easy to verify that

$$\begin{aligned} \hat{\rho}^{(k)}(u_{G_p C_p}, \bar{u}_{G_p C_p}) &= \max \left\{ \rho^{(0)}(u_{G_p C_p}, \bar{u}_{G_p C_p}), \dots, \rho^{(k)}(u_{G_p C_p}, \bar{u}_{G_p C_p}) \right\} \leq \\ &\leq e^{\mu_0} \max \left\{ \rho_\mu^{(0)}(u_{G_p C_p}, \bar{u}_{G_p C_p}), \dots, \rho_\mu^{(k)}(u_{G_p C_p}, \bar{u}_{G_p C_p}) \right\} = e^{\mu_0} \hat{\rho}_\mu^{(k)}(u_{G_p C_p}, \bar{u}_{G_p C_p}); \\ \hat{\rho}^{(k)}(W, \bar{W}) &= \max \left\{ \rho^{(0)}(W, \bar{W}), \dots, \rho^{(k)}(W, \bar{W}) \right\} \leq e^{\mu T_0} \max \left\{ \rho_\mu^{(0)}(W, \bar{W}), \dots, \rho_\mu^{(k)}(W, \bar{W}) \right\} = e^{\mu_0} \hat{\rho}_\mu^{(k)}(W, \bar{W}); \\ \hat{\rho}^{(k)}(J, \bar{J}) &= \max \left\{ \rho^{(0)}(J, \bar{J}), \dots, \rho^{(k)}(J, \bar{J}) \right\} \leq e^{\mu T_0} \max \left\{ \rho_\mu^{(0)}(J, \bar{J}), \dots, \rho_\mu^{(k)}(J, \bar{J}) \right\} = e^{\mu_0} \hat{\rho}_\mu^{(k)}(J, \bar{J}); \\ \rho_\mu^{(k)}(W, \bar{W}) &\leq (1/\mu) \rho_\mu^{(k)}(\dot{W}, \dot{\bar{W}}) \text{ and so on.} \end{aligned}$$

The set $M_0 \times M_W \times M_1 \times M_J$ turns out into a complete uniform space with respect to the family of pseudo-metrics

$$\begin{aligned} \hat{\rho}_\mu^{(k)}((u_{G_0 C_0}, W, u_{G_1 C_1}, J, \dot{u}_{G_0 C_0}, \dot{W}, \dot{u}_{G_1 C_1}, \dot{J}), (\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J}, \dot{\bar{u}}_{G_0 C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1 C_1}, \dot{\bar{J}})) = \\ = \max \left\{ \hat{\rho}^{(k)}(u_{G_0 C_0}, \bar{u}_{G_0 C_0}), \hat{\rho}^{(k)}(W, \bar{W}), \hat{\rho}^{(k)}(u_{G_1 C_1}, \bar{u}_{G_1 C_1}), \hat{\rho}^{(k)}(J, \bar{J}), \right. \\ \left. \hat{\rho}_\mu^{(k)}(\dot{u}_{G_0 C_0}, \dot{\bar{u}}_{G_0 C_0}), \hat{\rho}_\mu^{(k)}(\dot{W}, \dot{\bar{W}}), \hat{\rho}_\mu^{(k)}(\dot{u}_{G_1 C_1}, \dot{\bar{u}}_{G_1 C_1}), \hat{\rho}_\mu^{(k)}(\dot{J}, \dot{\bar{J}}) \right\}, (k=0,1,2,\dots). \end{aligned}$$

Remark 3.3 Replacing $t = t_0$ in (3.1) we obtain conformity condition (CC):

$$\begin{aligned}\dot{W}(t_0) &= -\dot{J}(0) + \left[Z_0 W(t_0) - Z_0 J(0) - 2\sqrt{L} u_{G_0 C_0}(t_0) + 2\sqrt{L} E_0(t_0) \right] / \left(d\tilde{L}_0(i(0, t_0)) / di_{G_0 C_0} \right), \\ \dot{J}(t_0) &= -\dot{W}(0) + \left[-Z_0 W(0) + Z_0 J(t_0) + 2\sqrt{L} u_{G_1 C_1}(t_0) - 2\sqrt{L} E_1(t_0) \right] / \left(d\tilde{L}_1(i(\Lambda, t_0)) / di_{G_1 C_1} \right)\end{aligned}$$

If $T = t_0$ and 0 are zero point of the initial functions and (without loss of generality) one can assume $u_{G_0 C_0}^T = 0$, $u_{G_1 C_1}^T = 0$ and (CC) becomes $\dot{W}(t_0) = -\dot{J}(0)$, $\dot{J}(t_0) = -\dot{W}(0)$. Then the assumptions $\dot{W}(t_0) = -\dot{J}(0) = 0$, $\dot{J}(t_0) = -\dot{W}(0) = 0$ imply (CC).

In order to avoid the problem with (CC) we define the operator

$$B = (B_0(u_{G_0 C_0}, W, u_{G_1 C_1}, J), B_W(u_{G_0 C_0}, W, u_{G_1 C_1}, J), B_1(u_{G_0 C_0}, W, u_{G_1 C_1}, J), B_J(u_{G_0 C_0}, W, u_{G_1 C_1}, J))$$

by the formulas

$$\begin{aligned}B_0^{(k)}(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) &= \int_{t_k}^t U_{G_0}(u_{G_0 C_0}, W, J)(s) ds - \frac{t - t_k}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} U_{G_0}(u_{G_0 C_0}, W, J)(s) ds, \quad t \in [t_k, t_{k+1}], \quad (k = 0, 1, 2, \dots), \\ B_W^{(k)}(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) &:= \int_{t_k}^t U(u_{G_0 C_0}, W, J)(s) ds - \frac{t - t_k}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} U(u_{G_0 C_0}, W, J)(s) ds, \quad t \in [t_k, t_{k+1}], \quad (k = 0, 1, 2, \dots), \\ B_1^{(k)}(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) &= \int_{t_k}^t U_{G_1}(W, u_{G_1 C_1}, J)(s) ds - \frac{t - t_k}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} U_{G_1}(W, u_{G_1 C_1}, J)(s) ds, \quad t \in [t_k, t_{k+1}], \quad (k = 0, 1, 2, \dots), \\ B_J^{(k)}(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) &:= \int_{t_k}^t I(W, u_{G_1 C_1}, J)(s) ds - \frac{t - t_k}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} I(W, u_{G_1 C_1}, J)(s) ds, \quad t \in [t_k, t_{k+1}], \quad (k = 0, 1, 2, \dots)\end{aligned}$$

where

$$\begin{aligned}U_{G_0}(u_{G_0 C_0}, W, J) &= \frac{1}{2\sqrt{L}} \frac{W(t) + \bar{J}(t) - 2\sqrt{L} G_0(u_{G_0 C_0}(t))}{d\tilde{C}_0(u_{G_0 C_0}) / du_{G_0 C_0}}, \\ U(u_{G_0 C_0}, W, J) &= -\frac{d\bar{J}(t)}{dt} + \frac{Z_0 W(t) - Z_0 \bar{J}(t) - 2\sqrt{L} u_{G_0 C_0}(t) + 2\sqrt{L} E_0(t)}{d\tilde{L}_0(i(0, t)) / di_{G_0 C_0}}, \\ U_{G_1}(W, u_{G_1 C_1}, J) &= \frac{1}{2\sqrt{L}} \frac{\bar{W}(t) + J(t) - 2\sqrt{L} G_1(u_{G_1 C_1}(t))}{d\tilde{C}_1(u_{G_1 C_1}) / du_{G_1 C_1}}, \\ I(W, u_{G_1 C_1}, J) &= -\frac{d\bar{W}(t)}{dt} + \frac{-Z_0 \bar{W}(t) + Z_0 J(t) + 2\sqrt{L} u_{G_1 C_1}(t) - 2\sqrt{L} E_1(t)}{d\tilde{L}_1(i(\Lambda, t)) / di_{G_1 C_1}}, \\ K(W): W \rightarrow \bar{W}(t) &= \begin{cases} \tilde{W}_0(t - T), & t \in [T, 2T] \\ W(t - T), & t \in [2T, \infty) \end{cases}, \quad K(J): J \rightarrow \bar{J}(t) = \begin{cases} \tilde{J}_0(t - T), & t \in [T, 2T] \\ J(t - T), & t \in [2T, \infty) \end{cases}.\end{aligned}$$

Lemma 3.1 The operator $K(\cdot)$ maps $C^1[t_0, \infty)$ into itself.

The proof is straightforward.

Lemma 3.2 If $(u_{G_0 C_0}, W, u_{G_1 C_1}, J) \in M_0 \times M_W \times M_1 \times M_J$ then

$$(B_0(t), B_W(t), B_1(t), B_J(t)) \in M_0 \times M_W \times M_1 \times M_J, \quad |\bar{W}(t)| \leq e^{-\beta} W_0 e^{\frac{R}{L} t}, \quad |\bar{J}(t)| \leq e^{-\beta} J_0 e^{\frac{R}{L} t}, \quad t \in [T, 2T].$$

Proof: We notice that $B_0(u_{G_0 C_0}, W, u_{G_1 C_1}, J)$ and $B_1(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t)$ are continuously differentiable functions, while $B_W(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t)$, $B_J(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t)$ are continuous ones. Indeed

$$\begin{aligned}\lim_{t \rightarrow t_{k+1} - 0} B_W^{(k)}(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) &= \int_{t_k}^{t_{k+1}} U(u_{G_0 C_0}, W, J)(s) ds - \frac{t_{k+1} - t_k}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} U(u_{G_0 C_0}, W, J)(s) ds = 0, \\ \lim_{t \rightarrow t_{k+1} + 0} B_W^{(k+1)}(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) &= \int_{t_{k+1}}^{t_{k+2}} U(u_{G_0 C_0}, W, J)(s) ds - \frac{t_{k+2} - t_{k+1}}{t_{k+2} - t_{k+1}} \int_{t_{k+1}}^{t_{k+2}} U(u_{G_0 C_0}, W, J)(s) ds = 0,\end{aligned}$$

$$\lim_{t \rightarrow t_{k+1}^-} B_J^{(k)}(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) = \int_{t_k}^{t_{k+1}} I(W, u_{G_1 C_1}, J)(s) ds - \frac{t_{k+1} - t_k}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} I(W, u_{G_1 C_1}, J)(s) ds = 0,$$

$$\lim_{t \rightarrow t_{k+1}^+} B_J^{(k+1)}(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) = \int_{t_{k+1}}^{t_{k+2}} I(W, u_{G_1 C_1}, J)(s) ds - \frac{t_{k+1} - t_{k+1}}{t_{k+2} - t_{k+1}} \int_{t_{k+1}}^{t_{k+2}} I(W, u_{G_1 C_1}, J)(s) ds = 0.$$

For the derivatives we obtain

$$\dot{B}_W(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) = \begin{cases} U(u_{G_0 C_0}, W, J)(t) - \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} U(u_{G_0 C_0}, W, J)(s) ds, t \in [t_k, t_{k+1}] \\ U(u_{G_0 C_0}, W, J)(t) - \frac{1}{t_{k+2} - t_{k+1}} \int_{t_{k+1}}^{t_{k+2}} U(u_{G_0 C_0}, W, J)(s) ds, t \in [t_{k+1}, t_{k+2}] \end{cases},$$

$$\dot{B}_J(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) = \begin{cases} I(W, u_{G_1 C_1}, J)(t) - \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} I(W, u_{G_1 C_1}, J)(s) ds, t \in [t_k, t_{k+1}] \\ I(W, u_{G_1 C_1}, J)(t) - \frac{1}{t_{k+2} - t_{k+1}} \int_{t_{k+1}}^{t_{k+2}} I(W, u_{G_1 C_1}, J)(s) ds, t \in [t_{k+1}, t_{k+2}] \end{cases}$$

and obviously $\dot{B}_W(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t)$, $\dot{B}_J(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t)$ are continuous on every interval (t_k, t_{k+1}) but not at t_k .

Lemma 3.2 is thus proved.

Lemma 3.3 Problem (3.1) has a solution $(u_{G_0 C_0}, W, u_{G_1 C_1}, J) \in M_0 \times M_W \times M_1 \times M_J$ if the operator B has a fixed point in $M_0 \times M_W \times M_1 \times M_J$, that is,

$$(u_{G_0 C_0}, W, u_{G_1 C_1}, J) = (B_0(u_{G_0 C_0}, W, u_{G_1 C_1}, J), B_W(u_{G_0 C_0}, W, u_{G_1 C_1}, J), B_1(u_{G_0 C_0}, W, u_{G_1 C_1}, J), B_J(u_{G_0 C_0}, W, u_{G_1 C_1}, J)). \quad (3.2)$$

Proof: Let $(u_{G_0 C_0}, W, u_{G_1 C_1}, J) \in M_0 \times M_W \times M_1 \times M_J$ be an oscillatory solution of (3.1). Then after integration of the equations (3.1) (recalling that $u_{R_0 L_0}(T) = u_{R_1 L_1}(T) = 0$) we obtain

$$u_{G_0 C_0}(t) = \int_{t_k}^t U_{G_0}(u_{G_0 C_0}, W, J)(s) ds, \quad W(t) = \int_{t_k}^t U(u_{G_0 C_0}, W, J)(s) ds, \quad u_{G_1 C_1}(t) = \int_{t_k}^t U_{G_1}(W, u_{G_1 C_1}, J)(s) ds, \quad J(t) = \int_{t_k}^t I(W, u_{G_1 C_1}, J)(s) ds$$

for $t \in [t_k, t_{k+1}]$, $(k = 0, 1, 2, \dots)$.

$$\text{Therefore } \int_{t_k}^{t_{k+1}} U(u_{G_0 C_0}, W, J)(s) ds = W(t_{k+1}) = 0, \quad \int_{t_k}^{t_{k+1}} I(W, u_{G_1 C_1}, J)(s) ds = J(t_{k+1}) = 0, \text{ which yields (3.2).}$$

Conversely, let B have a fixed point $(u_{G_0 C_0}, W, u_{G_1 C_1}, J) \in M_0 \times M_W \times M_1 \times M_J$. Then

$$\begin{aligned} \left| \int_{t_k}^{t_{k+1}} U_{G_0}(u_{G_0 C_0}, W, J)(s) ds \right| &\leq \left| \int_{t_k}^{t_{k+1}} \frac{W(s) + \bar{J}(s) - 2\sqrt{L}G_0(u_{G_0 C_0}(s))}{2\sqrt{L} \, d\tilde{C}_0(u_{G_0 C_0}(s)) / du_{G_0 C_0}} ds \right| \leq \\ &\leq \frac{1}{2\sqrt{L} \, \hat{C}_0^{(1)}} \left| W_0 \int_{t_k}^{t_{k+1}} e^{\mu(s-t_k)} ds + J_0 \int_{t_k}^{t_{k+1}} e^{\mu(s-t_k)} ds + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| \int_{t_k}^{t_{k+1}} |u_{G_0 C_0}(s)|^n ds \right| \leq \\ &\leq \frac{1}{2\sqrt{L} \, \hat{C}_0^{(1)}} \left| W_0 \int_{t_k}^{t_{k+1}} e^{\mu(s-t_k)} ds + J_0 \int_{t_k}^{t_{k+1}} e^{\mu(s-t_k)} ds + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \int_{t_k}^{t_{k+1}} e^{n\mu(s-t_k)} ds \right| \leq \\ &\leq \frac{1}{2\sqrt{L} \, \hat{C}_0^{(1)}} \left| W_0 \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} + J_0 \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \frac{e^{n\mu(t_{k+1}-t_k)} - 1}{n\mu} \right| \leq \\ &\leq \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} \frac{1}{2\sqrt{L} \, \hat{C}_0^{(1)}} \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n e^{(n-1)\mu(t_{k+1}-t_k)} \right) \leq \end{aligned}$$

$$\leq \frac{e^{\mu_0} - 1}{\mu} \frac{1}{2\sqrt{L} \hat{C}_0^{(1)}} \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n e^{(n-1)\mu_0} \right) \equiv M_{G_0}(\mu)$$

If we assume $\left| \int_{t_k}^{t_{k+1}} U_{G_0}(u_{G_0 C_0}, W, J)(s) ds \right| \neq 0$ then in view of $\lim_{\mu \rightarrow \infty} M_{G_0}(\mu) = 0$ we obtain a contradiction. It

follows $\left| \int_{t_k}^{t_{k+1}} U_{G_0}(u_{G_0 C_0}, W, J)(s) ds \right| = 0$. In analogous way we conclude

$$\left| \int_{t_k}^{t_{k+1}} U(u_{G_0 C_0}, W, J)(s) ds \right| = 0, \quad \left| \int_{t_k}^{t_{k+1}} U_{G_1}(W, u_{G_1 C_1}, J)(s) ds \right| = 0, \quad \left| \int_{t_k}^{t_{k+1}} I(W, u_{G_1 C_1}, J)(s) ds \right| = 0$$

Differentiating the above equalities we conclude that (3.1) has an oscillatory solution. Lemma 3.3 is thus proved.

IV. Existence-uniqueness of the Oscillatory pProblem

Theorem 4.1 Let the following conditions be fulfilled:

$$(E) \quad |E_p(s)| \leq U_{E_p} e^{-\frac{R}{L}s} \quad (p = 0, 1); \quad (L) \quad \tilde{L}_0(i) = \sum_{n=1}^m (n+1) l_n^{(0)}(i)^n \geq \hat{L}_0 > 0 \text{ for } |i| \leq I_0;$$

$$\text{Assumption } (\Pi): \quad \frac{W_0 + J_0}{2\sqrt{C}} \leq \phi_0, \quad U_{G_0}, U_{G_1} \leq \phi_0;$$

$$\text{Assumption (IN): } \left| \tilde{W}_0(t) \right| \leq W_0 e^{-\beta} e^{-\frac{R}{L}(t+T)}; \quad \left| \tilde{J}_0(t) \right| \leq J_0 e^{-\beta} e^{-\frac{R}{L}(t+T)}, \quad t \in [0, T].$$

Then there exists a unique oscillatory solution of (3.1).

Proof: We show that B maps $M_0 \times M_W \times M_1 \times M_J$ into itself. Indeed

$$\begin{aligned} \left| B_0(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) \right| &\leq \int_{t_k}^t \left| U(u_{G_0 C_0}, W, J)(s) \right| ds \leq \\ &\leq \frac{1}{2\hat{C}_0^{(1)}\sqrt{L}} \int_{t_k}^t \left| W_0 e^{-\frac{R}{L}s} + J_0 e^{-\frac{R}{L}(s+T)} - 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| u_{G_0 C_0}^n(s) \right| ds \leq \\ &\leq \frac{1}{2\hat{C}_0^{(1)}\sqrt{L}} \left(\int_{t_k}^t W_0 e^{-\frac{R}{L}s} ds + \int_{t_k}^t J_0 e^{-\frac{R}{L}(s+T)} ds + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| \int_{t_k}^t u_{G_0 C_0}^n(s) ds \right) \leq \\ &\leq \frac{1}{2\hat{C}_0^{(1)}\sqrt{L}} \left(W_0 \frac{L}{R} \left(e^{-\frac{R}{L}t_k} - e^{-\frac{R}{L}t} \right) + J_0 \frac{L}{R} e^{-\frac{R}{L}T} \left(e^{-\frac{R}{L}t_k} - e^{-\frac{R}{L}t} \right) + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \int_{t_k}^t e^{-\frac{R}{L}s} ds \right) \leq \\ &\leq \frac{1}{2\hat{C}_0^{(1)}\sqrt{L}} \left(\left(W_0 + J_0 e^{-\frac{R}{L}T} \right) \frac{L}{R} \left(e^{-\frac{R}{L}t_k} - e^{-\frac{R}{L}t} \right) + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \frac{L}{nR} \left(e^{-\frac{R}{L}t_k} - e^{-\frac{R}{L}t} \right) \right) \leq \\ &\leq \frac{1}{2\hat{C}_0^{(1)}\sqrt{L}} \frac{L}{R} \left(e^{-\frac{R}{L}t_k} - e^{-\frac{R}{L}t} \right) \left(W_0 + J_0 e^{-\frac{R}{L}T} + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \leq \\ &\leq \frac{e^{-\frac{R}{L}t}}{2\hat{C}_0^{(1)}\sqrt{L}} \frac{L}{R} \left(e^{\frac{R}{L}(t-t_k)} - 1 \right) \left(W_0 + J_0 e^{-\frac{R}{L}T} + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \leq \\ &\leq e^{-\frac{R}{L}t} \frac{1}{2\hat{C}_0^{(1)}\sqrt{L}} \frac{L}{R} \left(e^{\frac{RT_0}{L}} - e^{-\frac{RT_0}{L}} \right) \left(W_0 + J_0 e^{-\frac{R}{L}T} + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \leq \\ &\leq e^{-\frac{R}{L}t} \frac{1}{2\hat{C}_0^{(1)}\sqrt{L}} \frac{L}{R} \sinh\left(\frac{RT_0}{L}\right) \left(W_0 + J_0 e^{-\frac{R}{L}T} + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \leq e^{-\frac{R}{L}t} U_{G_0}. \end{aligned}$$

Further on we use the relations

$$\begin{aligned} \int_{t_k}^{t_{k+1}} e^{-\frac{R}{L}s} ds &= \frac{L}{R} \left(e^{-\frac{R}{L}t_k} - e^{-\frac{R}{L}t_{k+1}} \right) = \frac{L}{R} \left(e^{-\frac{R}{L}t_k} - e^{-\frac{R}{L}t} + e^{-\frac{R}{L}t} - e^{-\frac{R}{L}t_{k+1}} \right) = e^{-\frac{R}{L}t} \frac{L}{R} \left(e^{\frac{R}{L}(t-t_k)} - 1 + 1 - e^{-\frac{R}{L}(t_{k+1}-t)} \right) \leq \\ &\leq e^{-\frac{R}{L}t} \frac{L}{R} \left(e^{\frac{R}{L}(t_{k+1}-t_k)} - e^{-\frac{R}{L}(t_{k+1}-t_k)} \right) \leq e^{-\frac{R}{L}t} \frac{L}{R} \left(e^{\frac{R}{L}T_0} - e^{-\frac{R}{L}T_0} \right) = e^{-\frac{R}{L}t} \frac{2L}{R} \sinh\left(\frac{RT_0}{L}\right) \end{aligned}$$

and then we obtain

$$\left| B_W(u_{G_0C_0}, W, u_{G_1C_1}, J)(t) \right| \leq \left| \int_{t_k}^t U(u_{G_0C_0}, W, J)(s) ds \right| + \left| \frac{t-t_k}{t_{k+1}-t_k} \int_{t_k}^{t_{k+1}} U(u_{G_0C_0}, W, J)(s) ds \right| \equiv W_1 + W_2.$$

Since

$$\begin{aligned} W_1 &\leq \left| \int_{t_k}^t \frac{d\bar{J}(s)}{ds} ds \right| + \left| \int_{t_k}^t \frac{Z_0 W(s) - Z_0 \bar{J}(s) - 2\sqrt{L} u_{G_0C_0}(s) + 2\sqrt{L} E_0(s)}{\tilde{dL}_0(i(0,s))/di_{G_0C_0}} ds \right| \leq \\ &\leq |\bar{J}(t)| + \frac{1}{\hat{L}_0^{(1)}} \left(Z_0 \int_{t_k}^t |W(s)| ds + Z_0 \int_{t_k}^t |\bar{J}(s)| ds + 2\sqrt{L} \int_{t_k}^t |u_{G_0C_0}(s)| ds + 2\sqrt{L} \int_{t_k}^t |E_0(s)| ds \right) \leq \\ &\leq J_0 e^{-\beta} e^{-\frac{R}{L}(t+T)} + \frac{1}{\hat{L}_0^{(1)}} \left(Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0} \right) \int_{t_k}^t e^{-\frac{R}{L}s} ds \leq \\ &\leq J_0 e^{-\beta} e^{-\frac{R}{L}t} e^{-\frac{R}{L}T} + e^{-\frac{R}{L}t} \frac{1}{\hat{L}_0^{(1)}} \left(Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0} \right) \frac{L}{R} \left(e^{\frac{R}{L}T_0} - 1 \right) \leq \\ &\leq e^{-\frac{R}{L}t} \left[J_0 e^{-\beta} e^{-\frac{R}{L}T} + \frac{1}{\hat{L}_0^{(1)}} \left(Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0} \right) \frac{L}{R} \sinh\left(\frac{RT_0}{L}\right) \right] \end{aligned}$$

and

$$\begin{aligned} W_2 &\leq \left| \int_{t_k}^{t_{k+1}} \frac{d\bar{J}(s)}{ds} ds \right| + \left| \int_{t_k}^{t_{k+1}} \frac{Z_0 W(s) - Z_0 \bar{J}(s) - 2\sqrt{L} u_{G_0C_0}(s) + 2\sqrt{L} E_0(s)}{\tilde{dL}_0(i(0,s))/di_{G_0C_0}} ds \right| \leq \\ &\leq \frac{1}{\hat{L}_0^{(1)}} \left(Z_0 \int_{t_k}^{t_{k+1}} |W(s)| ds + Z_0 \int_{t_k}^{t_{k+1}} |\bar{J}(s)| ds + 2\sqrt{L} \int_{t_k}^{t_{k+1}} |u_{G_0C_0}(s)| ds + 2\sqrt{L} \int_{t_k}^{t_{k+1}} |E_0(s)| ds \right) \leq \\ &\leq \frac{Z_0 W_0 ds + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}}{\hat{L}_0^{(1)}} \int_{t_k}^{t_{k+1}} e^{-\frac{R}{L}s} ds \leq \\ &\leq e^{-\frac{R}{L}t} \frac{2L}{R} \sinh\left(\frac{RT_0}{L}\right) \frac{Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}}{\hat{L}_0^{(1)}} \end{aligned}$$

we have

$$\begin{aligned} \left| B_W(u_{G_0C_0}, W, u_{G_1C_1}, J)(t) \right| &\leq e^{-\frac{R}{L}t} \left[J_0 e^{-\beta} e^{-\frac{R}{L}T} + \frac{L}{R} \sinh\left(\frac{RT_0}{L}\right) \frac{Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}}{\hat{L}_0^{(1)}} \right] + \\ &+ e^{-\frac{R}{L}t} \frac{2L}{R} \sinh\left(\frac{RT_0}{L}\right) \frac{1}{\hat{L}_0^{(1)}} \left(Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0} \right) \leq \\ &\leq e^{-\frac{R}{L}t} \left[J_0 e^{-\beta} e^{-\frac{R}{L}T} + \frac{3L}{R} \sinh\left(\frac{RT_0}{L}\right) \frac{1}{\hat{L}_0^{(1)}} \left(Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0} \right) \right] \leq e^{-\frac{R}{L}t} W_0. \end{aligned}$$

Further on we have

$$\left| B_1(u_{G_0C_0}, W, u_{G_1C_1}, J)(t) \right| \leq \left| \int_{t_k}^t U_{G_1}(W, u_{G_1C_1}, J)(s) ds \right| \leq \int_{t_k}^t \frac{|\bar{W}(s) + J(s) - 2\sqrt{L} G_1(u_{G_1C_1}(s))|}{2\sqrt{L} dC_1(u_{G_1C_1})/du_{G_1C_1}} ds \leq$$

$$\begin{aligned}
 &\leq \frac{1}{2\hat{C}_1^{(1)}\sqrt{L}} \int_{t_k}^t \left| W_0 e^{-\frac{R}{L}(s+T)} + J_0 e^{-\frac{R}{L}s} - 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| |u_{G_1 C_1}(s)|^n \right| ds \leq \\
 &\leq \frac{1}{2\hat{C}_1^{(1)}\sqrt{L}} \left(\int_{t_k}^t W_0 e^{-\frac{R}{L}(s+T)} ds + \int_{t_k}^t J_0 e^{-\frac{R}{L}s} ds + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| \int_{t_k}^t |u_{G_1 C_1}(s)|^n ds \right) \leq \\
 &\leq \frac{1}{2\hat{C}_1^{(1)}\sqrt{L}} \left(W_0 e^{-\frac{R}{L}T} \frac{L}{R} \left(e^{-\frac{R}{L}t_k} - e^{-\frac{R}{L}t} \right) + J_0 \frac{L}{R} \left(e^{-\frac{R}{L}t_k} - e^{-\frac{R}{L}t} \right) + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \int_{t_k}^t e^{-n\frac{R}{L}s} ds \right) \leq \\
 &\leq \frac{1}{2\hat{C}_1^{(1)}\sqrt{L}} \left(\left(W_0 e^{-\frac{R}{L}T} + J_0 \right) \frac{L}{R} \left(e^{-\frac{R}{L}t_k} - e^{-\frac{R}{L}t} \right) + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \frac{L}{nR} \left(e^{-n\frac{R}{L}t_k} - e^{-n\frac{R}{L}t} \right) \right) \leq \\
 &\leq \frac{1}{2\hat{C}_1^{(1)}\sqrt{L}} \frac{L}{R} \left(e^{-\frac{R}{L}t_k} - e^{-\frac{R}{L}t} \right) \left(W_0 e^{-\frac{R}{L}T} + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \leq \\
 &\leq e^{-\frac{R}{L}t} \frac{1}{2\hat{C}_1^{(1)}\sqrt{L}} \frac{L}{R} \left(e^{\frac{R}{L}(t-t_k)} - 1 \right) \left(W_0 e^{-\frac{R}{L}T} + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \leq \\
 &\leq e^{-\frac{R}{L}t} \frac{1}{2\hat{C}_1^{(1)}\sqrt{L}} \frac{L}{R} \left(e^{\frac{RT_0}{L}} - e^{-\frac{RT_0}{L}} \right) \left(W_0 e^{-\frac{R}{L}T} + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \leq \\
 &\leq e^{-\frac{R}{L}t} \frac{1}{2\hat{C}_1^{(1)}\sqrt{L}} \frac{L}{R} \sinh\left(\frac{RT_0}{L}\right) \left(W_0 e^{-\frac{R}{L}T} + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \leq e^{-\frac{R}{L}t} U_{G_1}.
 \end{aligned}$$

For the last component we have

$$\left| B_J^{(k)}(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) \right| \leq \left| \int_{t_k}^t I(W, u_{G_1 C_1}, J)(s) ds \right| + \left| \frac{t-t_k}{t_{k+1}-t_k} \int_{t_k}^{t_{k+1}} I(W, u_{G_1 C_1}, J)(s) ds \right| \equiv I_1 + I_2.$$

But

$$\begin{aligned}
 I_1 &\leq \left| \int_{t_k}^t \frac{d\bar{W}(s)}{ds} ds \right| + \left| \int_{t_k}^t \frac{-Z_0 \bar{W}(s) + Z_0 J(s) + 2\sqrt{L} u_{G_1 C_1}(s) - 2\sqrt{L} E_1(s)}{\tilde{dL}_1(i(\Lambda, s)) / di_{G_1 C_1}} ds \right| \leq \\
 &\leq |\bar{W}(t)| + \frac{1}{\hat{L}_1^{(1)}} \left(Z_0 \int_{t_k}^t |\bar{W}(s)| ds + Z_0 \int_{t_k}^t |J(s)| ds + 2\sqrt{L} \int_{t_k}^t |u_{G_1 C_1}(s)| ds + 2\sqrt{L} \int_{t_k}^t |E_1(s)| ds \right) \leq \\
 &\leq W_0 e^{-\beta} e^{-\frac{R}{L}(t+T)} + \frac{1}{\hat{L}_1^{(1)}} \left(Z_0 W_0 e^{-\frac{R}{L}T} + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1} \right) \int_{t_k}^t e^{-\frac{R}{L}s} ds \leq \\
 &\leq W_0 e^{-\beta} e^{-\frac{R}{L}t} e^{-\frac{R}{L}T} + e^{-\frac{R}{L}t} \frac{1}{\hat{L}_1^{(1)}} \left(Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1} \right) \frac{L}{R} \left(e^{\frac{R}{L}T_0} - 1 \right) \leq \\
 &\leq e^{-\frac{R}{L}t} \left[W_0 e^{-\beta} e^{-\frac{R}{L}T} + \frac{1}{\hat{L}_1^{(1)}} \left(Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1} \right) \frac{L}{R} \sinh\left(\frac{R}{L}T_0\right) \right]
 \end{aligned}$$

and

$$\begin{aligned}
 I_2 &\leq \left| \int_{t_k}^{t_{k+1}} \frac{d\bar{W}(s)}{ds} ds \right| + \left| \int_{t_k}^{t_{k+1}} \frac{-Z_0 \bar{W}(s) + Z_0 J(s) - 2\sqrt{L} u_{G_1 C_1}(s) + 2\sqrt{L} E_1(s)}{\tilde{dL}_1(i(0, s)) / di_{G_1 C_1}} ds \right| \leq \\
 &\leq \frac{1}{\hat{L}_1^{(1)}} \left(Z_0 \int_{t_k}^{t_{k+1}} |\bar{W}(s)| ds + Z_0 \int_{t_k}^{t_{k+1}} |J(s)| ds + 2\sqrt{L} \int_{t_k}^{t_{k+1}} |u_{G_1 C_1}(s)| ds + 2\sqrt{L} \int_{t_k}^{t_{k+1}} |E_1(s)| ds \right) \leq
 \end{aligned}$$

$$\leq \frac{Z_0 W_0 e^{-\frac{R}{L}T} + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}}{\hat{L}_1^{(1)}} \int_{t_k}^{t_{k+1}} e^{-\frac{R}{L}s} ds \leq e^{-\frac{R}{L}t} \frac{2L}{R} \sinh\left(\frac{RT_0}{L}\right) \frac{Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}}{\hat{L}_1^{(1)}}.$$

Then

$$\left| B_f^{(k)}(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) \right| \leq e^{-\frac{R}{L}t} \left[W_0 e^{-\beta} e^{-\frac{R}{L}T} + \frac{3L}{R} \sinh\left(\frac{R}{L}T_0\right) \frac{Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}}{\hat{L}_1^{(1)}} \right] \leq e^{-\frac{R}{L}t} J_0.$$

It remains to obtain Lipschitz estimates for the components of the operator B .

Indeed, since $U_{G_0}(u_{G_0 C_0}, W, J) = \frac{1}{2\sqrt{L}} \frac{W(t) + \bar{J}(t) - 2\sqrt{L} G_0(u_{G_0 C_0}(t))}{d\tilde{C}_0(u_{G_0 C_0})/du_{G_0 C_0}}$ we get

$$\left| \frac{\partial U_{G_0}(u_{G_0 C_0}, W, J)}{\partial u_{G_0 C_0}} \right| = \frac{1}{2\sqrt{L}} \left| \frac{-2\sqrt{L} \frac{dG_0(u_{G_0 C_0}(t))}{du_{G_0 C_0}} \frac{d\tilde{C}_0(u_{G_0 C_0})}{du_{G_0 C_0}} - (W(t) + \bar{J}(t) - 2\sqrt{L} G_0(u_{G_0 C_0}(t))) \frac{d^2 \tilde{C}_0(u_{G_0 C_0})}{du_{G_0 C_0}^2}}{(d\tilde{C}_0(u_{G_0 C_0})/du_{G_0 C_0})^2} \right| \leq$$

$$\leq \frac{2\sqrt{L} \tilde{C}_0^{(1)} \sum_{n=1}^m n |g_n^{(0)}| U_{G_0}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \tilde{C}_0^{(2)}}{2\sqrt{L} (\hat{C}_0^{(1)})^2};$$

$$\left| \frac{\partial U_{G_0}(u_{G_0 C_0}, W, J)}{\partial W} \right| = \left| \frac{1}{2\sqrt{L} d\tilde{C}_0(u_{G_0 C_0})/du_{G_0 C_0}} \right| \leq \frac{1}{2\sqrt{L} \hat{C}_0^{(1)}};$$

$$\left| B_0(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) - B_0(\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J})(t) \right| = \left| \int_{t_k}^t U_{G_0}(u_{G_0 C_0}, W, J)(s) - U_{G_0}(\bar{u}_{G_0 C_0}, \bar{W}, \bar{J})(s) ds \right| \leq$$

$$\leq \int_{t_k}^t \left| \frac{\partial U_{G_0}}{\partial u_{G_0 C_0}} \right| |u_{G_0 C_0}(s) - \bar{u}_{G_0 C_0}(s)| ds + \int_{t_k}^t \left| \frac{\partial U_{G_0}}{\partial W} \right| |W(s) - \bar{W}(s)| ds + \int_{t_k}^t \left| \frac{\partial U_{G_0}}{\partial J} \right| |\bar{J}(s) - J(s)| ds \leq$$

$$\leq \int_{t_k}^t \left| \frac{\partial U_{G_0}}{\partial u_{G_0 C_0}} \right| |u_{G_0 C_0}(s) - \bar{u}_{G_0 C_0}(s)| ds + \int_{t_k}^t \left| \frac{\partial U_{G_0}}{\partial W} \right| |W(s) - \bar{W}(s)| ds \leq$$

$$\leq \frac{2\sqrt{L} \tilde{C}_0^{(1)} \sum_{n=1}^m n |g_n^{(0)}| U_{G_0}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \tilde{C}_0^{(2)}}{2\sqrt{L} (\hat{C}_0^{(1)})^2} \int_{t_k}^t |u_{G_0 C_0}(s) - \bar{u}_{G_0 C_0}(s)| ds +$$

$$+ \frac{1}{2\sqrt{L} \hat{C}_0^{(1)}} \int_{t_k}^t |W(s) - \bar{W}(s)| ds \leq$$

$$\leq \frac{2\sqrt{L} \tilde{C}_0^{(1)} \sum_{n=1}^m n |g_n^{(0)}| U_{G_0}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \tilde{C}_0^{(2)}}{2\sqrt{L} (\hat{C}_0^{(1)})^2} \rho_\mu^{(k)}(u_{G_0 C_0}, \bar{u}_{G_0 C_0}) \int_{t_k}^t e^{\mu(s-t_k)} ds +$$

$$+ \frac{1}{2\sqrt{L} \hat{C}_0^{(1)}} \rho_\mu^{(k)}(W, \bar{W}) \int_{t_k}^t e^{\mu(s-t_k)} ds \leq$$

$$\leq \frac{e^{\mu(t-t_k)} - 1}{\mu} \frac{2\sqrt{L} \tilde{C}_0^{(1)} \sum_{n=1}^m n |g_n^{(0)}| U_{G_0}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \tilde{C}_0^{(2)}}{2\sqrt{L} (\hat{C}_0^{(1)})^2} \rho_\mu^{(k)}(u_{G_0 C_0}, \bar{u}_{G_0 C_0}) +$$

$$+ \frac{e^{\mu(t-t_k)} - 1}{\mu} \frac{1}{2\sqrt{L} \hat{C}_0^{(1)}} \rho_\mu^{(k)}(W, \bar{W}) \leq$$

$$\begin{aligned} &\leq \frac{e^{\mu(t-t_k)} - 1}{\mu^2} \frac{2\sqrt{L}\tilde{C}_0^{(1)} \sum_{n=1}^m n |g_n^{(0)}| U_{G_0}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \tilde{C}_0^{(2)}}{2\sqrt{L}(\hat{C}_0^{(1)})^2} \rho_\mu^{(k)}(\dot{u}_{G_0C_0}, \dot{\bar{u}}_{G_0C_0}) + \\ &+ \frac{e^{\mu(t-t_k)} - 1}{\mu^2} \frac{1}{2\sqrt{L} \hat{C}_0^{(1)}} \rho_\mu^{(k)}(\dot{W}, \dot{\bar{W}}) \leq \\ &\leq \frac{e^{\mu(t-t_k)}}{\mu^2} \left[\frac{2\sqrt{L}\tilde{C}_0^{(1)} \sum_{n=1}^m n |g_n^{(0)}| U_{G_0}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \tilde{C}_0^{(2)}}{2\sqrt{L}(\hat{C}_0^{(1)})^2} + \frac{1}{2\sqrt{L} \hat{C}_0^{(1)}} \right] \times \\ &\times \hat{\rho}_\mu^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})) . \end{aligned}$$

It follows

$$\begin{aligned} &\hat{\rho}^{(k)}(B_0(u_{G_0C_0}, W, u_{G_1C_1}, J), B_0(\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J})) \leq \\ &\leq \frac{e^{\mu_0}}{\mu^2} \left[\frac{2\sqrt{L}\tilde{C}_0^{(1)} \sum_{n=1}^m n |g_n^{(0)}| U_{G_0}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \tilde{C}_0^{(2)}}{2\sqrt{L}(\hat{C}_0^{(1)})^2} + \frac{1}{2\sqrt{L} \hat{C}_0^{(1)}} \right] \times \\ &\times \hat{\rho}_\mu^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})) \equiv \\ &\equiv K_0 \hat{\rho}_\mu^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})) . \end{aligned}$$

Further on we have

$$\begin{aligned} &\left| \frac{\partial U(u_{G_0C_0}, W, J)}{\partial u_{G_0C_0}} \right| = \left| \frac{-2\sqrt{L} \frac{d\tilde{L}_0(i(0,t))}{di_{G_0C_0}} - (Z_0 W(t) - Z_0 \bar{J}(t) - 2\sqrt{L} u_{G_0C_0}(t) + 2\sqrt{L} E_0(t)) \frac{d\tilde{L}_0^2(i(0,t))}{di_{G_0C_0}^2}}{(d\tilde{L}_0(i(0,t)) / di_{G_0C_0})^2} \right| \leq \\ &\leq \frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} ; \\ &\left| \frac{\partial U(u_{G_0C_0}, W, J)}{\partial W} \right| = \left| \frac{Z_0}{d\tilde{L}_0(i(0,t)) / di_{G_0C_0}} \right| \leq \frac{Z_0}{\hat{L}_0^{(1)}} . \end{aligned}$$

But

$$\begin{aligned} &\left| B_W^{(k)}(u_{G_0C_0}, W, u_{G_1C_1}, J)(t) - B_W^{(k)}(\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J})(t) \right| \leq \\ &\leq \int_{t_k}^t |U(u_{G_0C_0}, W, J)(s) - U(\bar{u}_{G_0C_0}, \bar{W}, \bar{J})(s)| ds + \int_{t_k}^{t_{k+1}} |U(u_{G_0C_0}, W, J)(s) - U(\bar{u}_{G_0C_0}, \bar{W}, \bar{J})(s)| ds \equiv W_1 + W_2 ; \\ &W_1 \leq \int_{t_k}^t \left| \frac{\partial U(u_{G_0C_0}, W, J)}{\partial u_{G_0C_0}} \right| |u_{G_0C_0}(s) - \bar{u}_{G_0C_0}(s)| ds + \int_{t_k}^t \left| \frac{\partial U(u_{G_0C_0}, W, J)}{\partial W} \right| |W(s) - \bar{W}(s)| ds \leq \\ &\leq \frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} \int_{t_k}^t |u_{G_0C_0}(s) - \bar{u}_{G_0C_0}(s)| ds + \frac{Z_0}{\hat{L}_0^{(1)}} \int_{t_k}^t |W(s) - \bar{W}(s)| ds \leq \\ &\leq \frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} \rho_\mu^{(k)}(u_{G_0C_0}, \bar{u}_{G_0C_0}) \int_{t_k}^t e^{\mu(s-t_k)} ds + \\ &+ \frac{Z_0}{\hat{L}_0^{(1)}} \rho_\mu^{(k)}(W, \bar{W}) \int_{t_k}^t e^{\mu(s-t_k)} ds \leq \end{aligned}$$

$$\begin{aligned} &\leq \frac{e^{\mu(t-t_k)} - 1}{\mu^2} \frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} \rho_\mu^{(k)}(\dot{u}_{G_0 C_0}, \dot{\bar{u}}_{G_0 C_0}) + \\ &+ \frac{e^{\mu(t-t_k)} - 1}{\mu^2} \frac{Z_0}{\hat{L}_0^{(1)}} \rho_\mu^{(k)}(\dot{W}, \dot{\bar{W}}) \leq \\ &\leq \frac{e^{\mu(t-t_k)}}{\mu^2} \left(\frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} + \frac{Z_0}{\hat{L}_0^{(1)}} \right) \times \\ &\times \hat{\rho}_\mu^{(k)}((u_{G_0 C_0}, W, u_{G_1 C_1}, J, \dot{u}_{G_0 C_0}, \dot{W}, \dot{u}_{G_1 C_1}, \dot{J}), (\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J}, \dot{\bar{u}}_{G_0 C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1 C_1}, \dot{\bar{J}})) \end{aligned}$$

and

$$\begin{aligned} W_2 &\leq \int_{t_k}^{t_{k+1}} \left| \frac{\partial U(u_{G_0 C_0}, W, J)}{\partial u_{G_0 C_0}} \right| |u_{G_0 C_0}(s) - \bar{u}_{G_0 C_0}(s)| ds + \int_{t_k}^{t_{k+1}} \left| \frac{\partial U(u_{G_0 C_0}, W, J)}{\partial W} \right| |W(s) - \bar{W}(s)| ds \leq \\ &\leq \frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} \int_{t_k}^{t_{k+1}} |u_{G_0 C_0}(s) - \bar{u}_{G_0 C_0}(s)| ds + \frac{Z_0}{\hat{L}_0^{(1)}} \int_{t_k}^{t_{k+1}} |W(s) - \bar{W}(s)| ds \leq \\ &\leq \frac{e^{\mu(t_{k+1}-t_k)}}{\mu^2} \left(\frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} + \frac{Z_0}{\hat{L}_0^{(1)}} \right) \times \\ &\times \hat{\rho}_\mu^{(k)}((u_{G_0 C_0}, W, u_{G_1 C_1}, J, \dot{u}_{G_0 C_0}, \dot{W}, \dot{u}_{G_1 C_1}, \dot{J}), (\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J}, \dot{\bar{u}}_{G_0 C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1 C_1}, \dot{\bar{J}})) \end{aligned}$$

then

$$\begin{aligned} &|B_W^{(k)}(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) - B_W^{(k)}(\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J})(t)| \leq \\ &\leq \frac{2e^{\mu_0}}{\mu^2} \left(\frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} + \frac{Z_0}{\hat{L}_0^{(1)}} \right) \times \\ &\times \hat{\rho}_\mu^{(k)}((u_{G_0 C_0}, W, u_{G_1 C_1}, J, \dot{u}_{G_0 C_0}, \dot{W}, \dot{u}_{G_1 C_1}, \dot{J}), (\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J}, \dot{\bar{u}}_{G_0 C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1 C_1}, \dot{\bar{J}})) . \end{aligned}$$

It follows

$$\begin{aligned} &\hat{\rho}^{(k)}(B_W^{(k)}(u_{G_0 C_0}, W, u_{G_1 C_1}, J), B_W^{(k)}(\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J})) \leq \\ &\leq K_W \hat{\rho}_\mu^{(k)}((u_{G_0 C_0}, W, u_{G_1 C_1}, J, \dot{u}_{G_0 C_0}, \dot{W}, \dot{u}_{G_1 C_1}, \dot{J}), (\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J}, \dot{\bar{u}}_{G_0 C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1 C_1}, \dot{\bar{J}})) . \end{aligned}$$

For the third component we have

$$\begin{aligned} &|B_1(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) - B_1(\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J})(t)| \leq \int_{t_k}^t |U_{G_1}(W, u_{G_1 C_1}, J)(s) - U_{G_1}(\bar{W}, \bar{u}_{G_1 C_1}, \bar{J})(s)| ds \leq \\ &\leq \int_{t_k}^t \left| \frac{\partial U_{G_0}}{\partial u_{G_0 C_0}} \right| |u_{G_0 C_0}(s) - \bar{u}_{G_0 C_0}(s)| ds + \int_{t_k}^t \left| \frac{\partial U_{G_0}}{\partial W} \right| |W(s) - \bar{W}(s)| ds + \int_{t_k}^t \left| \frac{\partial U_{G_0}}{\partial J} \right| |J(s) - \bar{J}(s)| ds \leq \\ &\leq \int_{t_k}^t \left| \frac{\partial U_{G_1}}{\partial u_{G_1 C_1}} \right| |u_{G_1 C_1}(s) - \bar{u}_{G_1 C_1}(s)| ds + \int_{t_k}^t \left| \frac{\partial U_{G_1}}{\partial J} \right| |J(s) - \bar{J}(s)| ds \leq \\ &\leq \frac{2\sqrt{L} \tilde{C}_1^{(1)} \sum_{n=1}^m n |g_n^{(1)}| U_{G_1}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \tilde{C}_1^{(2)}}{2\sqrt{L} (\hat{C}_1^{(1)})^2} \int_{t_k}^t |u_{G_1 C_1}(s) - \bar{u}_{G_1 C_1}(s)| ds + \\ &+ \frac{1}{2\sqrt{L} \hat{C}_1^{(1)}} \int_{t_k}^t |J(s) - \bar{J}(s)| ds \leq \\ &\leq \frac{2\sqrt{L} \tilde{C}_1^{(1)} \sum_{n=1}^m n |g_n^{(1)}| U_{G_1}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \tilde{C}_1^{(2)}}{2\sqrt{L} (\hat{C}_1^{(1)})^2} \rho_\mu^{(k)}(u_{G_1 C_1}, \bar{u}_{G_1 C_1}) \int_{t_k}^t e^{\mu(s-t_k)} ds + \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2\sqrt{L} \hat{C}_1^{(1)}} \rho_\mu^{(k)}(J, \bar{J}) \int_{t_k}^t e^{\mu(s-t_k)} ds \leq \\
 & \leq \frac{e^{\mu(t-t_k)} - 1}{\mu} \frac{2\sqrt{L} \bar{C}_1^{(1)} \sum_{n=1}^m n |g_n^{(1)}| U_{G_1}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \bar{C}_1^{(2)}}{2\sqrt{L} (\hat{C}_1^{(1)})^2} \rho_\mu^{(k)}(u_{G_1 C_1}, \bar{u}_{G_1 C_1}) + \\
 & + \frac{e^{\mu(t-t_k)} - 1}{\mu} \frac{1}{2\sqrt{L} \hat{C}_1^{(1)}} \rho_\mu^{(k)}(J, \bar{J}) \leq \\
 & \leq \frac{e^{\mu(t-t_k)} - 1}{\mu^2} \frac{2\sqrt{L} \bar{C}_1^{(1)} \sum_{n=1}^m n |g_n^{(1)}| U_{G_1}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \bar{C}_1^{(2)}}{2\sqrt{L} (\hat{C}_1^{(1)})^2} \rho_\mu^{(k)}(\dot{u}_{G_1 C_1}, \dot{\bar{u}}_{G_1 C_1}) + \\
 & + \frac{e^{\mu(t-t_k)} - 1}{\mu^2} \frac{1}{2\sqrt{L} \hat{C}_1^{(1)}} \rho_\mu^{(k)}(J, \dot{\bar{J}}) \leq \\
 & \leq \frac{e^{\mu(t-t_k)}}{\mu^2} \left[\frac{2\sqrt{L} \bar{C}_1^{(1)} \sum_{n=1}^m n |g_n^{(1)}| U_{G_1}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \bar{C}_1^{(2)}}{2\sqrt{L} (\hat{C}_1^{(1)})^2} + \frac{1}{2\sqrt{L} \hat{C}_1^{(1)}} \right] \times \\
 & \times \hat{\rho}_\mu^{(k)}((u_{G_0 C_0}, W, u_{G_1 C_1}, J, \dot{u}_{G_0 C_0}, \dot{W}, \dot{u}_{G_1 C_1}, \dot{J}), (\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J}, \dot{\bar{u}}_{G_0 C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1 C_1}, \dot{\bar{J}})) .
 \end{aligned}$$

It follows

$$\begin{aligned}
 & \hat{\rho}^{(k)}(B_1(u_{G_0 C_0}, W, u_{G_1 C_1}, J), B_1(\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J})) \leq \\
 & \leq \frac{e^{\mu(t-t_k)}}{\mu^2} \left[\frac{2\sqrt{L} \bar{C}_1^{(1)} \sum_{n=1}^m n |g_n^{(1)}| U_{G_1}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \bar{C}_1^{(2)}}{2\sqrt{L} (\hat{C}_1^{(1)})^2} + \frac{1}{2\sqrt{L} \hat{C}_1^{(1)}} \right] \times \\
 & \times \hat{\rho}_\mu^{(k)}((u_{G_0 C_0}, W, u_{G_1 C_1}, J, \dot{u}_{G_0 C_0}, \dot{W}, \dot{u}_{G_1 C_1}, \dot{J}), (\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J}, \dot{\bar{u}}_{G_0 C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1 C_1}, \dot{\bar{J}})) \equiv \\
 & \equiv K_1 \hat{\rho}_\mu^{(k)}((u_{G_0 C_0}, W, u_{G_1 C_1}, J, \dot{u}_{G_0 C_0}, \dot{W}, \dot{u}_{G_1 C_1}, \dot{J}), (\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J}, \dot{\bar{u}}_{G_0 C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1 C_1}, \dot{\bar{J}})) .
 \end{aligned}$$

Finally for the fourth component we obtain

$$\begin{aligned}
 & \left| B_J^{(k)}(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) - B_J^{(k)}(\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J})(t) \right| \leq \int_{t_k}^t \left| I(W, u_{G_1 C_1}, J)(s) - I(\bar{W}, \bar{u}_{G_1 C_1}, \bar{J})(s) \right| ds + \\
 & + \left| \int_{t_k}^{t_{k+1}} \left(I(W, u_{G_1 C_1}, J)(s) - I(\bar{W}, \bar{u}_{G_1 C_1}, \bar{J})(s) \right) ds \right| \equiv I_1 + I_2 .
 \end{aligned}$$

Since

$$\begin{aligned}
 I_1 & \leq \int_{t_k}^t \left| \frac{\partial I(W, u_{G_1 C_1}, J)}{\partial u_{G_1 C_1}} \right| \left| u_{G_1 C_1}(s) - \bar{u}_{G_1 C_1}(s) \right| ds + \int_{t_k}^t \left| \frac{\partial I(W, u_{G_1 C_1}, J)}{\partial J} \right| \left| J(s) - \bar{J}(s) \right| ds \leq \\
 & \leq \frac{2\sqrt{L} \bar{L}_1^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}) \bar{L}_1^{(2)}}{(\hat{L}_1^{(1)})^2} \int_{t_k}^t \left| u_{G_1 C_1}(s) - \bar{u}_{G_1 C_1}(s) \right| ds + \frac{Z_0}{\hat{L}_1^{(1)}} \int_{t_k}^t \left| J(s) - \bar{J}(s) \right| ds \leq \\
 & \leq \frac{2\sqrt{L} \bar{L}_1^{(1)} + \left(Z_0 W_0 + Z_0 J_0 + \frac{Z_0}{\hat{L}_1^{(1)}} \rho_\mu^{(k)}(J, \bar{J}) \int_{t_k}^t e^{\mu(s-t_k)} ds \right)}{(\hat{L}_1^{(1)})^2} \leq 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1} \bar{L}_1^{(2)} \\
 & \leq \frac{2\sqrt{L} \bar{L}_1^{(1)} + \left(Z_0 W_0 + Z_0 J_0 + \frac{Z_0}{\hat{L}_1^{(1)}} \rho_\mu^{(k)}(J, \bar{J}) \int_{t_k}^t e^{\mu(s-t_k)} ds \right)}{(\hat{L}_1^{(1)})^2} \rho_\mu^{(k)}(u_{G_1 C_1}, \bar{u}_{G_1 C_1}) \int_{t_k}^t e^{\mu(s-t_k)} ds +
 \end{aligned}$$

$$\leq \frac{e^{\mu(t-t_k)} - 1}{\mu^2} \frac{2\sqrt{L} \tilde{L}_1^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}) \tilde{L}_1^{(2)}}{(\hat{L}_1^{(1)})^2} \rho_\mu^{(k)}(\dot{u}_{G_1 C_1}, \dot{\bar{u}}_{G_1 C_1}) + \frac{e^{\mu(t-t_k)} - 1}{\mu^2} \frac{Z_0}{\hat{L}_1^{(1)}} \rho_\mu^{(k)}(J, \dot{J}) \leq$$

$$\leq \frac{e^{\mu(t-t_k)}}{\mu^2} \left(\frac{2\sqrt{L} \tilde{L}_1^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}) \tilde{L}_1^{(2)}}{(\hat{L}_1^{(1)})^2} + \frac{Z_0}{\hat{L}_1^{(1)}} \right) \times$$

$$\times \hat{\rho}_\mu^{(k)}((u_{G_0 C_0}, W, u_{G_1 C_1}, J, \dot{u}_{G_0 C_0}, \dot{W}, \dot{u}_{G_1 C_1}, \dot{J}), (\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J}, \dot{\bar{u}}_{G_0 C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1 C_1}, \dot{\bar{J}}))$$

and

$$I_2 \leq \int_{t_k}^{t_{k+1}} \left| \frac{\partial I(W, u_{G_1 C_1}, J)}{\partial u_{G_1 C_1}} \right| |u_{G_1 C_1}(s) - \bar{u}_{G_1 C_1}(s)| ds + \int_{t_k}^{t_{k+1}} \left| \frac{\partial I(W, u_{G_1 C_1}, J)}{\partial J} \right| |J(s) - \bar{J}(s)| ds \leq$$

$$\leq \frac{e^{\mu(t_{k+1}-t_k)}}{\mu^2} \left(\frac{2\sqrt{L} \tilde{L}_1^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}) \tilde{L}_1^{(2)}}{(\hat{L}_1^{(1)})^2} + \frac{Z_0}{\hat{L}_1^{(1)}} \right) \times$$

$$\times \hat{\rho}_\mu^{(k)}((u_{G_0 C_0}, W, u_{G_1 C_1}, J, \dot{u}_{G_0 C_0}, \dot{W}, \dot{u}_{G_1 C_1}, \dot{J}), (\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J}, \dot{\bar{u}}_{G_0 C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1 C_1}, \dot{\bar{J}}))$$

we get

$$|B_J^{(k)}(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) - B_J^{(k)}(\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J})(t)| \leq$$

$$\leq \frac{2e^{\mu_0}}{\mu^2} \left(\frac{2\sqrt{L} \tilde{L}_1^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}) \tilde{L}_1^{(2)}}{(\hat{L}_1^{(1)})^2} + \frac{Z_0}{\hat{L}_1^{(1)}} \right) \times$$

$$\times \hat{\rho}_\mu^{(k)}((u_{G_0 C_0}, W, u_{G_1 C_1}, J, \dot{u}_{G_0 C_0}, \dot{W}, \dot{u}_{G_1 C_1}, \dot{J}), (\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J}, \dot{\bar{u}}_{G_0 C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1 C_1}, \dot{\bar{J}})) .$$

It follows

$$\hat{\rho}^{(k)}(B_J^{(k)}(u_{G_0 C_0}, W, u_{G_1 C_1}, J), B_J^{(k)}(\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J})) \leq$$

$$\leq K_J \hat{\rho}_\mu^{(k)}((u_{G_0 C_0}, W, u_{G_1 C_1}, J, \dot{u}_{G_0 C_0}, \dot{W}, \dot{u}_{G_1 C_1}, \dot{J}), (\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J}, \dot{\bar{u}}_{G_0 C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1 C_1}, \dot{\bar{J}})) .$$

For the derivatives we obtain

$$|\dot{B}_0(u_{G_0 C_0}, W, u_{G_1 C_1}, J)(t) - \dot{B}_0(\bar{u}_{G_0 C_0}, \bar{W}, \bar{u}_{G_1 C_1}, \bar{J})(t)| = |U_{G_0}(u_{G_0 C_0}, W, J)(s) - U_{G_0}(\bar{u}_{G_0 C_0}, \bar{W}, \bar{J})(s)| \leq$$

$$\leq \left| \frac{\partial U_{G_0}}{\partial u_{G_0 C_0}} \right| |u_{G_0 C_0}(t) - \bar{u}_{G_0 C_0}(t)| + \left| \frac{\partial U_{G_0}}{\partial W} \right| |W(t) - \bar{W}(t)| + \left| \frac{\partial U_{G_0}}{\partial J} \right| |J(t) - \bar{J}(t)| \leq$$

$$\leq \frac{2\sqrt{L} \tilde{C}_0^{(1)} \sum_{n=1}^m n |g_n^{(0)}| U_{G_0}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \tilde{C}_0^{(2)}}{2\sqrt{L} (\hat{C}_0^{(1)})^2} |u_{G_0 C_0}(t) - \bar{u}_{G_0 C_0}(t)| + \frac{|W(t) - \bar{W}(t)|}{2\sqrt{L} \hat{C}_0^{(1)}} \leq$$

$$\leq \frac{2\sqrt{L} \tilde{C}_0^{(1)} \sum_{n=1}^m n |g_n^{(0)}| U_{G_0}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \tilde{C}_0^{(2)}}{2\sqrt{L} (\hat{C}_0^{(1)})^2} e^{\mu(t-t_k)} \rho_\mu^{(k)}(u_{G_0 C_0}, \bar{u}_{G_0 C_0}) +$$

$$+ \frac{1}{2\sqrt{L} \hat{C}_0^{(1)}} e^{\mu(t-t_k)} \rho_\mu^{(k)}(W, \bar{W}) \leq$$

$$\leq \frac{e^{\mu(t-t_k)}}{\mu} \frac{2\sqrt{L} \tilde{C}_0^{(1)} \sum_{n=1}^m n |g_n^{(0)}| U_{G_0}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \tilde{C}_0^{(2)}}{2\sqrt{L} (\hat{C}_0^{(1)})^2} \rho_\mu^{(k)}(\dot{u}_{G_0 C_0}, \dot{\bar{u}}_{G_0 C_0}) +$$

$$+ \frac{e^{\mu(t-t_k)}}{\mu} \frac{1}{2\sqrt{L} \hat{C}_0^{(1)}} \rho_\mu^{(k)}(\dot{W}, \dot{\bar{W}}) \leq$$

$$\leq \frac{e^{\mu(t-t_k)}}{\mu} \left[\frac{2\sqrt{L} \tilde{C}_0^{(1)} \sum_{n=1}^m n |g_n^{(0)}| U_{G_0}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \tilde{C}_0^{(2)}}{2\sqrt{L} (\hat{C}_0^{(1)})^2} + \frac{1}{2\sqrt{L} \hat{C}_0^{(1)}} \right] \times$$

$$\times \hat{\rho}_{\mu}^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})) .$$

Then

$$\begin{aligned} & \rho_{\mu}^{(k)}(\dot{B}_0(u_{G_0C_0}, W, u_{G_1C_1}, J), \dot{B}_0(\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J})) \leq \\ & \leq \frac{1}{\mu} \left[\frac{2\sqrt{L} \tilde{C}_0^{(1)} \sum_{n=1}^m n |g_n^{(0)}| U_{G_0}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \tilde{C}_0^{(2)}}{2\sqrt{L} (\hat{C}_0^{(1)})^2} + \frac{1}{2\sqrt{L} \hat{C}_0^{(1)}} \right] \times \\ & \times \hat{\rho}_{\mu}^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})) \equiv \\ & \equiv \dot{K}_0 \hat{\rho}_{\mu}^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})). \end{aligned}$$

Further on we get

$$\begin{aligned} & |\dot{B}_W^{(k)}(u_{G_0C_0}, W, u_{G_1C_1}, J)(t) - \dot{B}_W^{(k)}(\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J})(t)| \leq \\ & \leq |U(u_{G_0C_0}, W, J)(t) - U(\bar{u}_{G_0C_0}, \bar{W}, \bar{J})(t)| + \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} |U(u_{G_0C_0}, W, J)(s) - U(\bar{u}_{G_0C_0}, \bar{W}, \bar{J})(s)| ds \equiv \dot{W}_1 + \dot{W}_2. \end{aligned}$$

But

$$\begin{aligned} \dot{W}_1 & \leq \left| \frac{\partial U(u_{G_0C_0}, W, J)}{\partial u_{G_0C_0}} \right| |u_{G_0C_0}(t) - \bar{u}_{G_0C_0}(t)| + \left| \frac{\partial U(u_{G_0C_0}, W, J)}{\partial W} \right| |W(t) - \bar{W}(t)| \leq \\ & \leq \frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} |u_{G_0C_0}(t) - \bar{u}_{G_0C_0}(t)| + \frac{Z_0}{\hat{L}_0^{(1)}} |W(t) - \bar{W}(t)| \leq \\ & \leq \frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} \rho_{\mu}^{(k)}(u_{G_0C_0}, \bar{u}_{G_0C_0}) e^{\mu(t-t_k)} + \frac{Z_0}{\hat{L}_0^{(1)}} \rho_{\mu}^{(k)}(W, \bar{W}) e^{\mu(t-t_k)} \leq \\ & \leq \frac{e^{\mu(t-t_k)}}{\mu} \frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} \rho_{\mu}^{(k)}(\dot{u}_{G_0C_0}, \dot{\bar{u}}_{G_0C_0}) + \frac{e^{\mu(t-t_k)}}{\mu} \frac{Z_0}{\hat{L}_0^{(1)}} \rho_{\mu}^{(k)}(\dot{W}, \dot{\bar{W}}) \leq \\ & \leq \frac{e^{\mu(t-t_k)}}{\mu} \left(\frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} + \frac{Z_0}{\hat{L}_0^{(1)}} \right) \times \\ & \times \hat{\rho}_{\mu}^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})) \end{aligned}$$

and

$$\begin{aligned} \dot{W}_2 & \leq \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \left| \frac{\partial U(u_{G_0C_0}, W, J)}{\partial u_{G_0C_0}} \right| |u_{G_0C_0}(s) - \bar{u}_{G_0C_0}(s)| ds + \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \left| \frac{\partial U(u_{G_0C_0}, W, J)}{\partial W} \right| |W(s) - \bar{W}(s)| ds \leq \\ & \leq \frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} |u_{G_0C_0}(s) - \bar{u}_{G_0C_0}(s)| ds + \\ & + \frac{Z_0}{\hat{L}_0^{(1)}} \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} |W(s) - \bar{W}(s)| ds \leq \\ & \leq \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu^2 (t_{k+1} - t_k)} \left(\frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} + \frac{Z_0}{\hat{L}_0^{(1)}} \right) \times \\ & \times \hat{\rho}_{\mu}^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})) \leq \\ & \leq \frac{1}{\mu} \frac{e^{\mu_0} - 1}{\mu_0} \left(\frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} + \frac{Z_0}{\hat{L}_0^{(1)}} \right) \times \\ & \times \hat{\rho}_{\mu}^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})). \end{aligned}$$

Therefore

$$\begin{aligned}
 & \left| \dot{B}_W^{(k)}(u_{G_0C_0}, W, u_{G_1C_1}, J)(t) - \dot{B}_W^{(k)}(\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J})(t) \right| \leq \\
 & \leq \frac{e^{\mu(t-t_k)}}{\mu} \left(\frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} + \frac{Z_0}{\hat{L}_0^{(1)}} \right) \times \\
 & \times \hat{\rho}_\mu^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})) + \\
 & + \frac{1}{\mu} \frac{e^{\mu_0} - 1}{\mu_0} \left(\frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} + \frac{Z_0}{\hat{L}_0^{(1)}} \right) \times \\
 & \times \hat{\rho}_\mu^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})) \leq \\
 & \leq e^{\mu(t-t_k)} \left(1 + \frac{e^{\mu_0} - 1}{\mu_0} \right) \frac{1}{\mu} \left(\frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{(\hat{L}_0^{(1)})^2} + \frac{Z_0}{\hat{L}_0^{(1)}} \right) \times \\
 & \times \hat{\rho}_\mu^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})) .
 \end{aligned}$$

It follows

$$\begin{aligned}
 & \hat{\rho}_\mu^{(k)}(B_W^{(k)}(u_{G_0C_0}, W, u_{G_1C_1}, J), B_W^{(k)}(\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J})) \leq \\
 & \leq \dot{K}_W \hat{\rho}_\mu^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})) .
 \end{aligned}$$

Since

$$\begin{aligned}
 & \left| \dot{B}_1(u_{G_0C_0}, W, u_{G_1C_1}, J)(t) - \dot{B}_1(\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J})(t) \right| \leq \left| U_{G_1}(W, u_{G_1C_1}, J)(s) - U_{G_1}(\bar{W}, \bar{u}_{G_1C_1}, \bar{J})(s) \right| \leq \\
 & \leq \left| \frac{\partial U_{G_1}}{\partial u_{G_1C_1}} \right| |u_{G_1C_1}(t) - \bar{u}_{G_1C_1}(t)| + \left| \frac{\partial U_{G_1}}{\partial J} \right| |J(t) - \bar{J}(t)| \leq \\
 & \leq \frac{2\sqrt{L} \tilde{C}_1^{(1)} \sum_{n=1}^m n |g_n^{(1)}| U_{G_1}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \tilde{C}_1^{(2)}}{2\sqrt{L} (\hat{C}_1^{(1)})^2} |u_{G_1C_1}(t) - \bar{u}_{G_1C_1}(t)| + \frac{1}{2\sqrt{L} \hat{C}_1^{(1)}} |J(t) - \bar{J}(t)| \leq \\
 & \leq e^{\mu(t-t_k)} \frac{2\sqrt{L} \tilde{C}_1^{(1)} \sum_{n=1}^m n |g_n^{(1)}| U_{G_1}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \tilde{C}_1^{(2)}}{2\sqrt{L} (\hat{C}_1^{(1)})^2} \rho_\mu^{(k)}(u_{G_1C_1}, \bar{u}_{G_1C_1}) + e^{\mu(t-t_k)} \frac{\rho_\mu^{(k)}(J, \bar{J})}{2\sqrt{L} \hat{C}_1^{(1)}} \leq \\
 & \leq e^{\mu(t-t_k)} \frac{1}{\mu} \frac{2\sqrt{L} \tilde{C}_1^{(1)} \sum_{n=1}^m n |g_n^{(1)}| U_{G_1}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \tilde{C}_1^{(2)}}{2\sqrt{L} (\hat{C}_1^{(1)})^2} \rho_\mu^{(k)}(\dot{u}_{G_1C_1}, \dot{\bar{u}}_{G_1C_1}) + e^{\mu(t-t_k)} \frac{1}{\mu} \frac{\rho_\mu^{(k)}(\dot{J}, \dot{\bar{J}})}{2\sqrt{L} \hat{C}_1^{(1)}} \leq \\
 & \leq \frac{e^{\mu(t-t_k)}}{\mu} \left[\frac{2\sqrt{L} \tilde{C}_1^{(1)} \sum_{n=1}^m n |g_n^{(1)}| U_{G_1}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \tilde{C}_1^{(2)}}{2\sqrt{L} (\hat{C}_1^{(1)})^2} + \frac{1}{2\sqrt{L} \hat{C}_1^{(1)}} \right] \times \\
 & \times \hat{\rho}_\mu^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}}))
 \end{aligned}$$

then

$$\begin{aligned}
 & \hat{\rho}_\mu^{(k)}(\dot{B}_1(u_{G_0C_0}, W, u_{G_1C_1}, J), \dot{B}_1(\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J})) \leq \\
 & \leq \frac{1}{\mu} \left[\frac{2\sqrt{L} \tilde{C}_1^{(1)} \sum_{n=1}^m n |g_n^{(1)}| U_{G_1}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \tilde{C}_1^{(2)}}{2\sqrt{L} (\hat{C}_1^{(1)})^2} + \frac{1}{2\sqrt{L} \hat{C}_1^{(1)}} \right] \times \\
 & \times \hat{\rho}_\mu^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})) \equiv \\
 & \equiv \dot{K}_1 \hat{\rho}_\mu^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})) .
 \end{aligned}$$

Finally for the fourth component we obtain

$$\left| \dot{B}_J^{(k)}(u_{G_0C_0}, W, u_{G_1C_1}, J)(t) - \dot{B}_J^{(k)}(\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J})(t) \right| \leq \left| I(W, u_{G_1C_1}, J)(t) - I(\bar{W}, \bar{u}_{G_1C_1}, \bar{J})(t) \right| +$$

$$+ \frac{1}{t_{k+1} - t_k} \left| \int_{t_k}^{t_{k+1}} (I(W, u_{G_1C_1}, J)(s) - I(\bar{W}, \bar{u}_{G_1C_1}, \bar{J})(s)) ds \right| \equiv \dot{I}_1 + \dot{I}_2.$$

Since

$$\dot{I}_1 \leq \left| \frac{\partial I(W, u_{G_1C_1}, J)}{\partial u_{G_1C_1}} \right| \left| u_{G_1C_1}(t) - \bar{u}_{G_1C_1}(t) \right| + \left| \frac{\partial I(W, u_{G_1C_1}, J)}{\partial J} \right| \left| J(t) - \bar{J}(t) \right| \leq$$

$$\leq e^{\mu(t-t_k)} \frac{2\sqrt{L} \tilde{L}_1^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}) \tilde{L}_1^{(2)}}{(\hat{L}_1^{(1)})^2} \rho_\mu^{(k)}(u_{G_1C_1}, \bar{u}_{G_1C_1}) + e^{\mu(t-t_k)} \frac{Z_0}{\hat{L}_1^{(1)}} \rho_\mu^{(k)}(J, \bar{J}) \leq$$

$$\leq \frac{e^{\mu(t-t_k)}}{\mu} \frac{2\sqrt{L} \tilde{L}_1^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}) \tilde{L}_1^{(2)}}{(\hat{L}_1^{(1)})^2} \rho_\mu^{(k)}(\dot{u}_{G_1C_1}, \dot{\bar{u}}_{G_1C_1}) + \frac{e^{\mu(t-t_k)}}{\mu} \frac{Z_0}{\hat{L}_1^{(1)}} \rho_\mu^{(k)}(\dot{J}, \dot{\bar{J}}) \leq$$

$$\leq \frac{e^{\mu(t-t_k)}}{\mu} \left(\frac{2\sqrt{L} \tilde{L}_1^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}) \tilde{L}_1^{(2)}}{(\hat{L}_1^{(1)})^2} + \frac{Z_0}{\hat{L}_1^{(1)}} \right) \times$$

$$\times \hat{\rho}_\mu^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})) ;$$

and

$$\dot{I}_2 \leq \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \left| \frac{\partial I(W, u_{G_1C_1}, J)}{\partial u_{G_1C_1}} \right| \left| u_{G_1C_1}(s) - \bar{u}_{G_1C_1}(s) \right| ds + \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \left| \frac{\partial I(W, u_{G_1C_1}, J)}{\partial J} \right| \left| J(s) - \bar{J}(s) \right| ds \leq$$

$$\leq \frac{2\sqrt{L} \tilde{L}_1^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}) \tilde{L}_1^{(2)}}{(\hat{L}_1^{(1)})^2} \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \left| u_{G_1C_1}(s) - \bar{u}_{G_1C_1}(s) \right| ds +$$

$$+ \frac{Z_0}{\hat{L}_1^{(1)}} \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \left| J(s) - \bar{J}(s) \right| ds \leq$$

$$\leq \frac{1}{\mu} \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu(t_{k+1} - t_k)} \left(\frac{2\sqrt{L} \tilde{L}_1^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}) \tilde{L}_1^{(2)}}{(\hat{L}_1^{(1)})^2} + \frac{Z_0}{\hat{L}_1^{(1)}} \right) \times$$

$$\times \hat{\rho}_\mu^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}}))$$

we obtain

$$\left| \dot{B}_J^{(k)}(u_{G_0C_0}, W, u_{G_1C_1}, J)(t) - \dot{B}_J^{(k)}(\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J})(t) \right| \leq$$

$$\leq \left(1 + \frac{e^{\mu_0} - 1}{\mu_0} \right) \frac{1}{\mu} \left(\frac{2\sqrt{L} \tilde{L}_1^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}) \tilde{L}_1^{(2)}}{(\hat{L}_1^{(1)})^2} + \frac{Z_0}{\hat{L}_1^{(1)}} \right) \times$$

$$\times \hat{\rho}_\mu^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})).$$

Thus

$$\hat{\rho}_\mu^{(k)}(B_J^{(k)}(u_{G_0C_0}, W, u_{G_1C_1}, J), B_J^{(k)}(\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J})) \leq$$

$$\leq \dot{K}_J \hat{\rho}_\mu^{(k)}((u_{G_0C_0}, W, u_{G_1C_1}, J, \dot{u}_{G_0C_0}, \dot{W}, \dot{u}_{G_1C_1}, \dot{J}), (\bar{u}_{G_0C_0}, \bar{W}, \bar{u}_{G_1C_1}, \bar{J}, \dot{\bar{u}}_{G_0C_0}, \dot{\bar{W}}, \dot{\bar{u}}_{G_1C_1}, \dot{\bar{J}})).$$

Therefore if $K = \max\{K_{R_0L_0}, K_u, \dot{K}_{R_0L_0}, \dot{K}_u\} < 1$ then the operator B has a unique fixed point. It is an oscillatory solution of the problem stated.

Theorem 4.1 is thus proved.

V. Numerical example

We collect all inequalities guaranteeing an existence-uniqueness of an oscillatory solution:

$$\frac{1}{\hat{C}_0^{(1)}} \frac{L}{\sqrt{L}} \frac{R}{R} \sinh\left(\frac{RT_0}{L}\right) \left(W_0 + J_0 e^{-\frac{R}{L} T} + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \leq U_{G_0} ;$$

$$\left[J_0 e^{-\beta} e^{-\frac{R}{L} T} + \frac{3L}{R} \sinh\left(\frac{R}{L} T_0\right) \frac{Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}}{\hat{L}_0^{(1)}} \right] \leq W_0 ;$$

$$\frac{1}{\hat{C}_1^{(1)} \sqrt{L}} \frac{L}{R} \sinh\left(\frac{RT_0}{L}\right) \left(W_0 e^{-\frac{R}{L} T} + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \leq U_{G_1} ;$$

$$\left[W_0 e^{-\beta} e^{-\frac{R}{L} T} + \frac{3L}{R} \sinh\left(\frac{R}{L} T_0\right) \frac{Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}}{\hat{L}_1^{(1)}} \right] \leq J_0 ;$$

$$K_0 = \frac{e^{\mu_0}}{2\sqrt{L} \hat{C}_0^{(1)} \mu^2} \left[\frac{2\sqrt{L} \tilde{C}_0^{(1)} \sum_{n=1}^m n |g_n^{(0)}| U_{G_0}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) \tilde{C}_0^{(2)}}{\hat{C}_0^{(1)}} + 1 \right] < 1 ;$$

$$K_W = \frac{2e^{\mu_0}}{\hat{L}_0^{(1)} \mu^2} \left(\frac{2\sqrt{L} \tilde{L}_0^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{\hat{L}_0^{(1)}} + Z_0 \right) < 1 ;$$

$$K_1 = \frac{e^{\mu_0}}{2\sqrt{L} \hat{C}_1^{(1)} \mu^2} \left[\frac{2\sqrt{L} \tilde{C}_1^{(1)} \sum_{n=1}^m n |g_n^{(1)}| U_{G_1}^{n-1} + \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) \tilde{C}_1^{(2)}}{\hat{C}_1^{(1)}} + 1 \right] < 1 ;$$

$$K_J = \frac{2e^{\mu_0}}{\hat{L}_1^{(1)} \mu^2} \left(\frac{2\sqrt{L} \tilde{L}_1^{(1)} + (Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}) \tilde{L}_1^{(2)}}{\hat{L}_1^{(1)}} + Z_0 \right) < 1 ;$$

$$\dot{K}_0 = \frac{1}{2\hat{C}_0^{(1)} \sqrt{L} \mu} \left[2\sqrt{L} \frac{\tilde{C}_0^{(1)}}{\hat{C}_0^{(1)}} \sum_{n=1}^m n |g_n^{(0)}| U_{G_0}^{n-1} + \frac{\tilde{C}_0^{(2)}}{\hat{C}_0^{(1)}} \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(0)}| U_{G_0}^n \right) + 1 \right] < 1 ;$$

$$\dot{K}_W = \left(1 + \frac{e^{\mu_0} - 1}{\mu_0} \right) \frac{1}{\hat{L}_0^{(1)} \mu} \left(\frac{2\sqrt{L} \tilde{L}_0^{(1)}}{\hat{L}_0^{(1)}} + \frac{(Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_0} + 2\sqrt{L} U_{E_0}) \tilde{L}_0^{(2)}}{\hat{L}_0^{(1)}} + Z_0 \right) < 1 ;$$

$$\dot{K}_1 = \frac{1}{2\hat{C}_1^{(1)} \sqrt{L} \mu} \left[2\sqrt{L} \frac{\tilde{C}_1^{(1)}}{\hat{C}_1^{(1)}} \sum_{n=1}^m n |g_n^{(1)}| U_{G_1}^{n-1} + \frac{\tilde{C}_1^{(2)}}{\hat{C}_1^{(1)}} \left(W_0 + J_0 + 2\sqrt{L} \sum_{n=1}^m |g_n^{(1)}| U_{G_1}^n \right) + 1 \right] < 1 ;$$

$$\dot{K}_J = \left(1 + \frac{e^{\mu_0} - 1}{\mu_0} \right) \frac{1}{\hat{L}_1^{(1)} \mu} \left(\frac{2\sqrt{L} \tilde{L}_1^{(1)}}{\hat{L}_1^{(1)}} + \frac{(Z_0 W_0 + Z_0 J_0 + 2\sqrt{L} U_{G_1} + 2\sqrt{L} U_{E_1}) \tilde{L}_1^{(2)}}{\hat{L}_1^{(1)}} + Z_0 \right) < 1 .$$

Consider a transmission line with length $\Lambda = 100m$, cross-section area $S = 4mm^2$, specific resistance for the cuprum $\rho_c = 0,0175$. Then $R = (\rho_c \Lambda) / S \approx 0,44\Omega$. Let $L = 0,45\mu H/m$; $C = 80pF/m$; $v = 1/\sqrt{LC} = 1/(6.10^{-9})$; $Z_0 = \sqrt{L/C} = 75\Omega$; $T = \Lambda\sqrt{LC} = 10^4.6.10^{-9} = 6.10^{-5}$. Let us check the propagation of waves with length $\lambda_0 = (1/6)10^{-4}m$. Then $f_0 = 1/(\lambda_0\sqrt{LC}) = 10^{13}Hz \Rightarrow T_0 = 1/f_0 = 10^{-13}$. Choose $\mu = 10^{14}$, then $\mu T_0 = \mu_0 = 10$, and $T = 6.10^8.T_0 \Rightarrow m = 6.10^8$; $(RT)/L \approx 58,7$; $e^{-(RT)/L} = e^{-58,7} \approx 0$; $(RT_0)/L \approx 0,98.10^{-7}$; $\sinh(RT_0/L) \approx RT_0/L = 0,98.10^{-7}$; $(L/R)\sinh(RT_0/L) \approx T_0$. If $G_0(i) = G_1(i) = 0,028i - 0,125i^3$ and $L_0(i) = L_1(i) = 3i - (1/12)i^3$ then

$$d\tilde{L}_0(i)/di = i(dL_0(i)/di) + L_0(i) = 6i - (1/3)i^3; \quad d^2\tilde{L}_0(i)/di^2 = 6 - i^2.$$

Choose $I_0 = 1$ then

$$d\tilde{L}_0(i)/di = i - (1/3)i^3 > 6 - (1/3) = \hat{L}_0^{(1)} = 17/3 \text{ and } |d\tilde{L}_0(i)/di| \leq 6 + 1/3 \Rightarrow \tilde{L}_0^{(1)} = 19/3, |d^2\tilde{L}_0(i)/di^2| \leq \tilde{L}_0^{(2)} = 7$$

.

Let us take $C_0(u) = C_1(u) = c_0 \sqrt{\Phi_0} / \sqrt{\Phi_0 - u}$, where $h = 2$, $c_0 = 50 \text{ pF} = 5 \cdot 10^{-11} \text{ F}$ and $\Phi_0 = 0,4 \text{ V}$; $\phi_0 = 0,04$.

Then $C_0(u) = c_0 \sqrt{\Phi_0} / \sqrt{\Phi_0 - u} \geq C_0(-\phi_0) = c_0 \sqrt{\Phi_0} / \sqrt{\Phi_0 + \phi_0} \Rightarrow \hat{C}_0 = 5 \cdot 10^{-11} \cdot \sqrt{0,4} / \sqrt{0,4 + 0,2} = 3,8 \cdot 10^{-11}$;

$$\left| \frac{d\tilde{C}_0(u)}{du} \right| \leq \tilde{C}_0^{(1)} = \frac{c_0(1 + 0,5(\phi_0 / \Phi_0))}{[1 - (\phi_0 / \Phi_0)]^{\frac{3}{2}}} \approx c_0 = 5 \cdot 10^{-11};$$

$$\frac{d\tilde{C}_0(u)}{du} \geq \hat{C}_0^{(1)} = \frac{c_0}{(1 + (\phi_0 / \Phi_0))^{\frac{3}{2}}} \left(1 - \frac{h-1}{h} (\phi_0 / \Phi_0) \right) \approx c_0 = 5 \cdot 10^{-11};$$

$$\left| \frac{d^2\tilde{C}_0(u)}{du^2} \right| \leq \tilde{C}_0^{(2)} = \frac{c_0 \sqrt[3]{\Phi_0}}{h^2} \frac{2h^2\Phi_0 + (2h^2 + h + 1)\phi_0}{(\Phi_0 - \phi_0)^{2 + \frac{1}{h}}} \approx 2,36 \cdot 10^{-9}.$$

Then for $W_0 = J_0 = 10^{-5}$; $U_{G_0} = 0,01$; $U_{E_0} = 0,01$ the above inequalities become: $2,98(10^{-5} + 0,038 \cdot 10^{-5}) \leq 10^{-2}$;

$9 \cdot 10^{-13} (150 \cdot 10^{-5} + 2 \cdot 10^{-5} \sqrt{0,45 \cdot 2}) / 17 \leq 10^{-5}$; $\dot{K}_0 = 4,5 \cdot 10^{-4} < 1$;

$$\dot{K}_W = \frac{388,6}{10^{14}} \left(\frac{8,5 \cdot 10^{-3} + (15 \cdot 10^{-4} + 0,027 \cdot 10^{-3})7}{5,6666} + 75 \right) \approx \frac{2,9145}{10^{14}}.$$

VI. Conclusion

We have noticed that the transmission line loaded by the shown configurations of nonlinear elements (cf. Figure 1) requires different way of derivation of the boundary conditions and respectively leads to different type of differential equations. Here one more difficulty arises. One could not exclude some transitional currents and so we have to solve a system of 4 equations for 4 unknown functions. Our fixed point method, however, is again applicable and we obtain existence-uniqueness of an oscillatory solution for lossy lines.

We point out that the procedure for treating of time-varying specific parameters might be applied to the present circuit.

VII. References

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