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Abstract: In this study, using the properties of first and second chebyshev wavelets \( \psi_1'(t) \) and \( \psi_2'(t) \) respectively, we explicitly present the relationship between them. Then a new formula expressing the first derivative of first kind Chebyshev wavelets \( \frac{d\psi_1(t)}{dt} \) in terms of \( \psi_2(t) \), and a formula expressing the derivative of second kind Chebyshev wavelets \( \frac{d\psi_2(t)}{dt} \) in terms of \( \psi_1(t) \) and \( \psi_2(t) \) is deduced. The relation between the first derivatives of \( \psi_2(t) \) is also given in this paper. All the proposed results are of direct interest in many applications.

Keywords: first chebyshev wavelets, \( \psi_1'(t) \), second chebyshev wavelets \( \psi_2'(t) \), operational matrix of derivative.

I. Introduction

Wavelets theory is a relatively new emerging in mathematical research [1-4]. It has been applied in a wide range of engineering disciplines, particularly, wavelets play an important role in establishing algebraic methods for the solution of integral differential equations [5,6], analysis of time-varying or time delay system and optimal control [7-9]. This paper is organized as follow: section 2 talks about the first and second shifted Chebyshev wavelets. Section 3, gives some important properties of first and second Chebyshev wavelets, the relation between \( \psi_1'(t) \) and \( \psi_2(t) \), the operational matrices of derivative of \( \psi_1'(t) \) and \( \psi_2(t) \). Section 4, discusses some important relationship between the operational matrices of derivative for \( \psi_1'(t) \) and \( \psi_2(t) \) in terms of \( \psi_1'(t) \) and \( \psi_2(t) \) themselves.

II. Chebyshev Wavelets

Wavelets constitute a family of signal functions constructed from dilation called the mother wavelet when the dilation parameter \( a \) and translation parameter \( b \) vary continuously, we have the following family of wavelets [5].

\[
\Psi_{a,b}(t) = |a|^{1/2}\psi\left(\frac{t-b}{a}\right) \quad a, b \in \mathbb{R} \quad a \neq 0
\]

where \( \Psi(t) = [\Psi_0(t), \Psi_1(t), ..., \Psi_{M-1}(t)]^T \)

The elements \( \Psi_0(t), \Psi_1(t), ..., \Psi_{M-1}(t) \) are the basis functions, orthogonal on the [0, 1]. We should note in dealing with Chebyshev wavelets the weight function \( w(t) \) have to dilated and translated as

\[
w_n(t) = w(2^k t - 2n + 1)
\]

Chebyshev polynomials are encountered in several areas of numerical analysis, and they hold particular importance in various subjects such as orthogonal polynomials, polynomials approximation, numerical integration and spectral methods [10-13]. Depending on the definition of Chebyshev polynomials we can present the diffinitions of Chebyshev wavelets for first and second kinds[3,6].

Definition (1) Shifted First Chebyshev Wavelets

first Chebyshev wavelets \( \Psi_{n,m}^1 = \Psi_{k,n,m,t}^1 \) \( k,n,m,t \) have four arguments; \( n \) argument \( k \) can assume any positive integer and \( t \) is the normalized time. They are defined on the interval \([0,1]\) by

\[
\Psi_{n,m}^1(t) = \begin{cases} \frac{2^k t - 2n + 1}{2^k - 1}T_m(2^k t - 2n - 1) & \frac{n}{2^k} \leq t < \frac{n+1}{2^k} \\ 0 & \text{otherwise} \end{cases}
\]

where \( T_m = \begin{cases} \frac{1}{\sqrt{n}}T_m & m = 0 \\ \frac{2}{\sqrt{n}}T_m & m \geq 1 \end{cases} \)

\( m = 0, 1, ..., M \) and \( n = 0, 1, 2, ..., 2^k - 1 \)

Definition (2) Shifted Second Chebyshev Wavelets \( \Psi_{n,m}^2 \).
Second Chebyshev wavelets $\Psi_{n,m}^2 = \Psi_{k,n,m,t}^2 \Psi_{k,n,m,t}^2$ have four arguments; n argument k can assume any positive integer and t is the normalized time. They are defined on the interval $[0,1)$ by

$$\Psi_{n,m}^2(t) = \begin{cases} \sum_{k=0}^{n} U_m(2^{k-1}t - 2n - 1) & 2k-1 \leq t < \frac{n+1}{2k-1} \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (2)

where $m = 0, 1, 2, \ldots, M$, $n = 0, 1, 2, \ldots, 2^{k-1} - 1$ and $U_m(t) = \frac{2}{\sqrt{\pi}} \int_0^t J_{2k-1}(2\pi U_m) dt$

A function $f(t)$ defined over $L^2[0,1]$ can be expanded in terms of $\Psi_{nm}^2(t)$ either $\Psi_{t}^2(t)$ or $\Psi_{2t}^2(t)$

$$f(t) = \sum_{n=0}^\infty \sum_{m=0}^\infty C_{nm} \Psi_{nm}^2(t)$$  \hspace{1cm} (3)

where $C_{nm} = \langle f(t), \Psi_{nm}^2(t) \rangle$

If infinite series in (3) is truncated then it can be written as

$$f(t) = \sum_{n=0}^{2k-1} \sum_{m=0}^{M-1} C_{nm} \Psi_{nm}^2(t)$$

where $C_{nm} = \langle f(t), \Psi_{nm}^2(t) \rangle$

C and $\Psi_{nm}^2(t)$ are $2^k(M+1) \times 1$ matrices.

**III. New Relation between $\Psi_{nm}^1$ and $\Psi_{nm}^2$.**

For

$$\Psi_{n,m}^2 - \Psi_{n,m-2}^2 = \frac{2\sqrt{2}}{c_m} \Psi_{n,m}^1(t)$$  \hspace{1cm} (6)

where $c_m = \left\{ \begin{array}{ll} \sqrt{2} & m = 0 \\ 2 & m = 1, 2, \ldots \end{array} \right.$  \hspace{1cm} (7)

From the following relation between $T_m$, $U_m$;

$$U_m(x) - U_{m-2}(x) = 2 T_m(x), \ m = 2, 3, \ldots$$

we have

$$\Psi_{n,m}^2 = \sqrt{2} \sum_{k=0}^{k-1} (2^{k+1} - 2n + 1)$$

$$\Psi_{n,m}^2 = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{k-1} (2^{k+1} - 2n + 1)$$

or $\Psi_{n,m}^2 - \Psi_{n,m-2}^2 = \sqrt{2} \sum_{k=0}^{k-1} (2^{k+1} - 2n + 1)$ then $\Psi_{n,m}^2 - \Psi_{n,m-2}^2 = \frac{2\sqrt{2}}{c_m} \Psi_{n,m}^1(t)$, $m = 2, 3, \ldots$

**IV. Operational Matrices of Derivative for $\Psi_{nm}^1$ and $\Psi_{nm}^2$.**

**Theorem (1):**

Let $\Psi_{t}^1(t)$ be Chebyshev wavelet vectors defined by

$$\Psi_{t}^1(t) = \left[ \Psi_{00}^1, \Psi_{01}^1, \ldots, \Psi_{M0}^1, \Psi_{01}^1, \ldots, \Psi_{(2k-1)0}^1, \Psi_{(2k-1)1}^1, \ldots, \Psi_{(2k-1)M}^1 \right]^T$$

The derivative of this vector $\Psi_{t}^1(t)$ can be expressed by

$$\frac{d\Psi_{t}^1}{dt} = D_{\Psi_{t}^1} \Psi_{t}^1(t)$$  \hspace{1cm} (8)

where $D_{\Psi_{t}^1}$ is the $2^k(M+1)$ matrix of derivative defined as follows

$$D_{\Psi_{t}^1} = \left[ \begin{array}{cccc} F & 0 & \cdots & 0 \\ 0 & F & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F \end{array} \right]$$

In which $F$ is $(M+1) \times (M+1)$ matrix and its $(r, s)$ element is defined as follows

$$F_{rs} = \left\{ \begin{array}{ll} 2k+1 & r = 2, \ldots, (M+1), s = 1, \ldots, r-1 \\ 0 & \text{otherwise} \end{array} \right.$$  \hspace{1cm} (9)

**Theorem (2):**

Let $\Psi_{2t}^2(t)$ be the second Chebyshev wavelet vectors defined by

$$\Psi_{2t}^2(t) = \left[ \Psi_{00}^2, \Psi_{01}^2, \ldots, \Psi_{M0}^2, \Psi_{01}^2, \ldots, \Psi_{(2k-1)0}^2, \Psi_{(2k-1)1}^2, \ldots, \Psi_{(2k-1)M}^2 \right]^T$$

then the derivative of $\Psi_{2t}^2(t)$ can be expressed by

$$\frac{d\Psi_{2t}^2}{dt} = D_{\Psi_{2t}^2} \Psi_{2t}^2(t)$$

where $D_{\Psi_{2t}^2}$ is the $2^{k-1}(M+1)$ operational matrix of derivative defined as follows
In which $L$ is $(M + 1) \times (M + 1)$ matrix and its $(r, s)$ the element is defined as follow

$$L_{r,s} = \begin{cases} (2k+1)^s & \text{for } r=2,3,...(M+1) \text{ and } s=1,2,...,(r-1) \\ 0 & \text{otherwise} \end{cases}$$

(11)

**Proof:**

By using shifted second Chebyshev polynomials in to $[0,1]$ the $r$th element of vector $\Psi^2(t)$ can be written as

$$\Psi^2_r(t) = \left\{ \begin{array}{ll} 2^k \sqrt{\frac{2}{\pi}} \sum_{m=0}^{n-1} U_m^s(2^k t-n) & t \in \left[ \frac{n}{2^k-1}, \frac{n-1}{2^k-1} \right] \\ 0 & \text{otherwise} \end{array} \right.$$  

(12)

where $r = n(M+1) + (m+1) \quad m = 0,1,\ldots,M$ and $n = 0,1,\ldots,(2^{k-1} - 1)$

Differentiation equation (12) w.r.t $t$ we have

$$\frac{d\Psi^2_r}{dt} = \left\{ \begin{array}{ll} 2^k \sqrt{\frac{2}{\pi}} \sum_{m=0}^{n-1} U_m^s(2^k t-n) & n \leq t < \frac{n+1}{2^k-1} \\ 0 & \text{otherwise} \end{array} \right.$$  

(13)

that is $\Psi^2_r(t), i = n(M+1) + 1, n(M+1) + 2,\ldots,(n+1)(M+1)$ so, its second Chebyshev wavelets expansion has the following form

$$\frac{d\Psi^2_r}{dt} = \sum_{i=n(M+1)+1}^{(n+1)(M+1)} a_i \Psi^2_i$$

This implies that the operational matrix $D_{\Psi^2}$ is a block matrix as defined in (10), Moreover; we have $\frac{dU^r(t)}{dt} = 0$

This results that $\frac{d\Psi^2_r}{dt} = 0$ for $r = 1, (M+1) + 1,\ldots,(2^{k-1} - 1)(M+1) + 1$.

Consequently the first row of matrix $L$ defined in (11) is zero.

with the aid of the first derivative of shifted second kind Chebyshev polynomial given by the following formula

$$U_m = \sum_{k=0}^{m-1} 4(k+1)U^r_k, \quad m = 1,2,\ldots,M + k \text{ odd}$$

$$\frac{dU^r}{dt} = 2^k \sqrt{\frac{2}{\pi}} \sum_{m=0}^{n-1} U_m^s(2^k t-n)$$  

(14)

$$\frac{dU^r}{dt} = 2^k \sum_{m=0}^{n-1} U_m^s(2^k t-n)$$  

(15)

where $r = 2,\ldots,(M+1), s = 1,\ldots,r = 1$ and $r = (r + s) \text{ odd}

(16)

Equation $\frac{d\Psi^2}{dt} = D_{\Psi^2}$ is hold.

**V. Some New Relationships between the Derivatives of $\Psi^1_{nm}$ and $\Psi^2_{nm}$**

Some new relationships between $\Psi^1_{nm}(t)$ and $\Psi^2_{nm}(t)$ are derived and given throughout the following lemmas.

**Lemma (1)**

The first derivative of $\Psi^1_{nm}(t)$ in terms of $\Psi^2_{nm}(t)$ is formulated as

$$\frac{d\Psi^1_{nm}(t)}{dt} = 2^{k+1} m \sum_{r=1}^{r-1} \Psi^2_{n(M+1)+s}$$  

(17)

**Proof**

By using shifted first Chebyshev polynomials in to $[0,1], \Psi^1_{nm}(t)$ can be written as

$$\Psi^1_{nm}(t) = \sum_{k=0}^{n-1} \sum_{s=0}^{r-1} \frac{c_{r-1}^s \Psi^1_{n(M+1)+s}(t)}{4 \sqrt{t^2 - 2^k t - 2n - 1}}$$

Then differentiation with respect to $t$ yields,

$$\frac{d\Psi^1_{nm}(t)}{dt} = 2^{k+1} m \sum_{r=1}^{r-1} \Psi^2_{n(M+1)+s}(t)$$  

(18)

where $r = 2,\ldots,M + 1$
Since $\Psi_{r-1}^1(t) = \frac{2}{\sqrt{\pi}}$ then $\frac{d\Psi_{r-1}^1(t)}{dt} = 2^{k+1}m \sqrt{Z} \Psi_{r-1}^1$

That is $\frac{d\Psi_{r-1}^1(t)}{dt} = 2^{k+1}m \Psi_{r-1}^2(t)$

Finally if $\frac{2^{r-1}}{c_{s-1}} = 1$ Then equation (18) becomes $\frac{d\Psi_{r-1}^1(t)}{dt} = 2^{k+1}m \sum_{s=1}^{r} \Psi_{n(M+1)+s}^2(t)$

which is the required result.

**Lemma 3:**
The derivative of $\Psi_{r-1}^2(t)$ in terms of $\Psi_{r}^1(t)$ and $\Psi_{r-2}^2(t)$ is formulated as

$$\frac{d\Psi_{r}^2(t)}{dt} = \left(2^{k+1}\sum_{s=1}^{r} \frac{2^{r-s}}{c_{s-1}} \Psi_{n(M+1)+s}^1(t) + \sum_{s=1}^{r-1} \Psi_{n(M+1)+s-2}^2(t)\right)$$

(19) where $r = 2, ..., (M+1), s = 1, 2, ..., r-1$ and $c_{s-1} = \left\{\begin{array}{ll} \sqrt{Z} & \text{s} - 1 = 0 \\ 2 & \text{otherwise} \end{array}\right.$

**Proof:**
From theorem (2)

$$\frac{d\Psi_{r}^2(t)}{dt} = 2^{k+1} \sum_{s=1}^{r} \Psi_{n(M+1)+s}^2(t) \quad s + r = \text{odd} \quad (20)$$

and from relationships between $\Psi_{nm}^1$ and $\Psi_{nm}^2$ in chapter 2 and substitute it in (20) we obtained following equation

$$\frac{d\Psi_{r}^2(t)}{dt} = \left(2^{k+1} \sum_{s=1}^{r} \frac{2^{r-s}}{c_{s-1}} \Psi_{n(M+1)+s}^1(t) + \sum_{s=1}^{r-1} \Psi_{n(M+1)+s-2}^2(t)\right)$$

or

$$\frac{d\Psi_{r}^2(t)}{dt} = \left(2^{k+1} \sum_{s=1}^{r} \frac{2^{r-s}}{c_{s-1}} \Psi_{n(M+1)+s}^1(t) + \sum_{s=1}^{r-1} \Psi_{n(M+1)+s-2}^2(t)\right)$$

then equation (20) is hold.

**Lemma (3)**
The first derivative of $\Psi_{r-1}^2(t)$ in terms of the second derivative of $\Psi_{r}^1(t)$ is given by the formula:

$$\frac{d\Psi_{r}^2(t)}{dt} = \frac{1}{2^{k+1}m} \sum_{s=1}^{r} \Psi_{n(M+1)+s+1}^1(t) \quad (r + s) \text{ odd}$$

(21)

**Proof:**
From lemma (1)

$$\frac{d\Psi_{r}^1(t)}{dt} = (2^{k+1}m) \sum_{s=1}^{r-1} \Psi_{n(M+1)+s}^2(t)$$

Differentiation above equation to obtain

$$\frac{d^2\Psi_{r}^1(t)}{dt^2} = (2^{k+1}m) \sum_{s=1}^{r-1} \frac{d\Psi_{n(M+1)+s}^2(t)}{dt}$$

Divided both sides $(2^{k+1}m)$

$$\frac{1}{2^{k+1}m} \frac{d^2\Psi_{r}^1(t)}{dt^2} = \sum_{s=1}^{r-1} \frac{d\Psi_{n(M+1)+s}^2(t)}{dt}$$

$$r - 1 = n(M+1) + s$$

$$r = n(M+1) + s + 1$$

then

$$\frac{d\Psi_{r}^2(t)}{dt} = \frac{1}{2^{k+1}m} \sum_{s=1}^{r} \Psi_{n(M+1)+s+1}^1(t) \quad r = 3, ..., (M+1), s = 1, ..., (r - 1)$$

which is the required result.

VI. Conclusion

In this work, at first, we demonstrate the relation between Chebyshev wavelets of the first and second kinds. Then we derived some important relationships between Chebyshev wavelets and their operational matrices of derivative, which plays an important role for algebraic methods.

VII. References


