A Common Fixed point Theorem for Three Self-Mappings in Cone Metric Space

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Abstract: The aim of this paper is to prove a coincidence and common fixed point theorem of three self-mappings satisfying contractive type condition (A) in cone metric space.

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I. Introduction and Preliminaries

Huang & Zhang [2] familiarized the idea of cone metric space and prove some fixed point theorems for contractive type mappings in a normal cone metric space. Subsequently, some other authors [1, 3 to 6] studied the existence of fixed points of self-mappings satisfying a contractive type condition. Here, we obtain points of coincidence and common fixed points for three self-mappings satisfying condition (A) in a complete cone metric space.

Definition 1.1 (see [1]): A subset $P$ of a real Banach space $E$ is called a cone if it has the following properties:

1. $P$ is non-empty, closed and $P \neq \{0\}$
2. $0 \leq a, b \in \mathbb{R}$ and $u, v \in P \Rightarrow au + bv \in P$;
3. $u \in P$ and $-u \in P \Rightarrow u = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Definition 1.2 (see [1]): For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ on $E$ with respect to $P$ by $u \leq v$ if and only if $u - v \in P$. We shall write $u < v$ if $u \leq v$ while $u \ll v$ stands for $v - u \in P^0$ where $P^0$ denotes the interior of $P$. The cone $P$ is said normal if for some $K > 0$ for all $u, v \in E$,

$$0 \leq u \leq v \Rightarrow \|u\| \leq K\|v\|$$

(1.1)

The least positive number $K$ satisfying (1.1) is called the normal constant of $P$.

In the following, we always suppose that $E$ is a real Banach space and $P$ is a cone in $E$ with $\text{int}P \neq 0$ and $\leq$ is a partial ordering with respect to $P$.

Proposition 1.3 (see [7]): Let $P$ be a cone in a real Banach space $E$. If for $b \in P$ and $b \geq \alpha b$, for some $\alpha \in [0, 1)$ then $b = 0$.

Proposition 1.4 (see [7]): Let $P$ be a cone in a real Banach space $E$ with non-empty interior. If for $b \in E$ and $b \ll c$, for all $c \in P^0$, then $b = 0$.

Definition 1.5 (see [1]): Let $X$ be a nonempty set. Suppose that the mapping $d: X \times X \to E$ satisfies

1. $0 \leq d(u, v), \forall u, v \in X$ and $d(u, v) = 0 \Leftrightarrow u = v$;
2. $d(u, v) = d(v, u), \forall u, v \in X$;
3. $d(u, v) \leq d(u, w) + d(w, v), \forall u, v, w \in X$. 
Then $d$ is called a cone metric on $X$, and $(X, d)$ is called a cone metric space.

**Example 1.6** (see [1]): Let $X = \mathbb{R}^2$, $P = \{(u, v) \in X : u, v \geq 0\} \subset \mathbb{R}^2$, $d:X \times X \to E$ such that $d((u, v)) = (\|u - v\|, \alpha|u - v|)$, where $\alpha \geq 0$ is a constant. Then $(X, d)$ is cone metric space.

**Definition 1.7** (see [1]): Let $\{u_n\}$ be a sequence in $X$ and $u \in X$. If for each $0 < c, \exists n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(u_n, u) < c$, then $\{u_n\}$ is said to be convergent (or $\{u_n\}$ converges) to $u$ and $u$ is called the limit of $\{u_n\}$. We denote this by $\lim_{n \to \infty} u_n = u$ or $u_n \to u$ as $n \to \infty$. If for each $0 < c \exists n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(u_n, u_m) < c$, then $\{u_n\}$ is called a Cauchy sequence in $X$. If every Cauchy sequence is convergent in $X$, then $X$ is called a complete cone metric space.

**Definition 1.8** (see [1]): A pair $(f, T)$ of self-mappings on $X$ one said to be weakly compatible if they commute at their coincidence point i.e. $fTu = Tfu$ whenever $fu = Tu$.

**Definition 1.9** (see [1]): A point $v \in X$ is called a point of coincidence of $f$ and $T$ if $\exists$ a point $u \in X$ such that $v = fu = Tu$.

**Condition (A):** Let $(X, d)$ be a cone metric space, $P$ be a normal cone with normal constant $K$ and $S, T, f : X \to X$ are three self-mappings. Then $S, T, f$ are said to satisfy condition (A) if

$$d(Su, Tv) \leq ad(fv, Sw) \left| \frac{1 + d(fu, Tu)}{1 + d(fu, fu)} \right| + b[d(fu, Su) + d(fv, Tv)] + cd(fu, fv)$$

for $u, v \in X$ where $0 < a, b, c < 1$ with $a + 2b + c < 1$.

**Proposition 1.10** (see [6]): Let $(X, d)$ be a cone metric space and $P$ be a cone in a real Banach space $E$. If $u \leq v, v \leq w$ then $u \leq w$.

**II. Main Results**

**Theorem 2.1:** Let $(X, d)$ be a cone metric space, $P$ be a normal cone with normal constant $K$. Suppose the mapping $S, T, f : X \to X$ satisfying condition (A). If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of $X$, then $S, T$ and $f$ have a unique point of coincidence. Moreover, if $(S, f)$ and $(T, f)$ are weakly compatible, then $S, T$ and $f$ have a unique common fixed point.

**Proof:** Let $u_0$ be arbitrary point in $X$. Choose a point $u_1$ in $X$ such that $fu_0 = Su_1$. Similarly, choose a point $u_2$ in $X$ such that $fu_2 = Tu_1$. Continuing in this way choose $u_k$ in $X$ to obtain $u_{k+1}$ in $X$ such that

$$fu_{2k+1} = Su_{2k}, \quad fu_{2k+2} = Tu_{2k+1}, \quad (k \geq 0)$$

Then,

$$d(fu_{2k+1}, fu_{2k+2}) = d(Su_{2k}, Tu_{2k+1}) \leq ad(fu_{2k+1}, Su_{2k+1}) \left[ \frac{1 + d(fu_{2k+1}, Tu_{2k+1})}{1 + d(fu_{2k+1}, fu_{2k+1})} \right]$$

$$+ b[d(fu_{2k}, Su_{2k}) + d(fu_{2k+1}, Tu_{2k+1})] + cd(fu_{2k}, fu_{2k+1})$$

$$\leq ad(fu_{2k+1}, fu_{2k+2}) \left[ \frac{1 + d(fu_{2k+1}, fu_{2k+1})}{1 + d(fu_{2k}, fu_{2k+1})} \right]$$

$$+ b[d(fu_{2k}, fu_{2k+1}) + d(fu_{2k+1}, fu_{2k+1})] + cd(fu_{2k}, fu_{2k+1})$$

This $\Rightarrow$

$$d(fu_{2k+1}, fu_{2k+2}) \leq \left[ \frac{b + c}{1 - (a + b)} \right] d(fu_{2k}, fu_{2k+1})$$

Similarly,
Now for positive integer \( n \) write
\[
\lambda f_{u_{2k+2}, f_{u_{2k+3}}} = d(S_{u_{2k+1}}, T_{u_{2k+2}}) \\
\leq d(f_{u_{2k+2}, S_{u_{2k+2}}}) \left[ 1 + d(f_{u_{2k+1}, T_{u_{2k+2}}}) \right]
\]
\[+ b [d(f_{u_{2k+1}, S_{u_{2k+1}}} + d(f_{u_{2k+2}, T_{u_{2k+2}}}) + cd(f_{u_{2k+1}}, f_{u_{2k+2}})
\leq d(f_{u_{2k+2}}, f_{u_{2k+3}}) \left[ 1 + d(f_{u_{2k+1}, f_{u_{2k+2}}}) \right]
\]
\[+ b [d(f_{u_{2k+1}}, f_{u_{2k+2}} + d(f_{u_{2k+2}}, f_{u_{2k+3}})] + cd(f_{u_{2k+1}}, f_{u_{2k+2}})
\]

This \( \Rightarrow \)

\[d(f_{u_{2k+2}}, f_{u_{2k+3}}) \leq \left[ \frac{b+c}{1-(a+b)} \right] d(f_{u_{2k+1}}, f_{u_{2k+2}})
\]

Now by induction we arrive at

\[d(f_{u_{2k+3}}, f_{u_{2k+2}}) \leq \left[ \frac{b+c}{1-(a+b)} \right]^2 d(f_{u_{2k}}, f_{u_{2k+1}})
\]
\[\leq \ldots \leq \left[ \frac{b+c}{1-(a+b)} \right]^k d(f_{u_0}, f_{u_1})
\]

And

\[d(f_{u_{2k+2}}, f_{u_{2k+3}}) \leq \left[ \frac{b+c}{1-(a+b)} \right]^2 d(f_{u_{2k+1}}, f_{u_{2k+2}})
\]
\[\leq \ldots \leq \left[ \frac{b+c}{1-(a+b)} \right]^{k+1} d(f_{u_0}, f_{u_1})
\]

for each \( k \geq 0. \)

Let \( = \frac{b+c}{1-(a+b)}, \) then \( \lambda < 1. \)

Hence in general we can write
\[d(f_{u_k}, f_{u_{k+1}}) \leq \lambda^k d(f_{u_0}, f_{u_1})
\]

Now for positive integer \( p, \) we have
\[d(f_{u_k}, f_{u_{k+p}}) \leq d(f_{u_k}, f_{u_{k+1}}) + d(f_{u_{k+1}}, f_{u_{k+2}}) + \ldots + d(f_{u_{k+p-1}}, f_{u_{k+p}})
\]
\[\leq [\lambda^k + \lambda^{k+1} + \lambda^{k+2} + \ldots + \lambda^{k+p-1}] d(f_{u_0}, f_{u_1})
\]
\[\leq \lambda^k \left[ \frac{1-\lambda^{1-p}}{1-\lambda} \right] d(f_{u_0}, f_{u_1})
\]
\[\leq \lambda^k \left[ \frac{1}{1-\lambda} \right] d(f_{u_0}, f_{u_1})
\]

Now for \( c \in P^0, \exists \ r > 0 \) such that \( c - v \in P^0 \) if \( \|v\| < r. \) Choose a positive integer \( n_0 \) such that for all \( k \geq n_0, \left\| \lambda^k d(f_{u_0}, f_{u_1}) \right\| < r, \) which implies that,
\( c - \frac{\delta k}{1+\delta} d(f u_{2k}, f u_1) \in P^0 \) and \( \frac{\delta k}{1+\delta} d(f u_{2k}, f u_k) \in P^0 \)

\( k > N_0 \) and for all \( P \), by Proposition 1.10, \( d(f u_{2k}, f u_{k+1}) \ll c \) for all \( k > N_0 \) and for all \( P \). Hence \( \{ f u_k \} \) is a Cauchy sequence in \( f(X) \). Since \( f(X) \) is complete, there exist \( x, y \in X \) such that \( f u_k \to y = f x \).

Since

\[
d(f(x, Sx)) \leq d(f(x, f u_{2k})) + d(f u_{2k}, Sx)
\]

\[
\leq d(y, f u_{2k}) + d(T u_{2k-1}, Sx)
\]

\[
\leq d(y, f u_{2k}) + d(Sx, T u_{2k-1})
\]

\[
\leq d(y, f u_{2k}) + d(f u_{2k-1}, S u_{2k-1}) \left[ \frac{1+d(f(x, T x))}{1+d(f(x, f u_{2k-1}))} \right]
\]

\[
+ b d(f(x, S x)) + d(f u_{2k-1}, T u_{2k-1}) + c d(f(x, f u_{2k-1}))
\]

\[
= d(y, f u_{2k}) + d(f u_{2k-1}, f u_{2k}) \left[ \frac{1+d(f(x, T x))}{1+d(f(x, f u_{2k-1}))} \right]
\]

\[
+ b d(f(x, S x)) + d(f u_{2k-1}, f u_{2k}) + c d(f(x, f u_{2k-1}))
\]

This \( \Rightarrow \)

\[
d(f(x, Sx)) \leq \frac{1}{1-b} \left[ d(y, f u_{2k}) + d(f u_{2k-1}, f u_{2k}) \left[ \frac{1+d(f(x, T x))}{1+d(f(x, f u_{2k-1}))} \right]
\]

\[
+ b d(f u_{2k-1}, f u_{2k}) + c d(f(x, f u_{2k-1})) \right]
\]

Hence, it concludes that

\[
\|d(f(x, Sx))\| \leq \frac{K}{1-b} \left[ d(y, f u_{2k}) + d(f u_{2k-1}, f u_{2k}) \left[ \frac{1+d(f(x, T x))}{1+d(f(x, f u_{2k-1}))} \right]
\]

\[
+ b d(f u_{2k-1}, f u_{2k}) + c d(f(x, f u_{2k-1})) \right]
\]

where \( K \) is a normal constant. If \( n \to \infty \), then we arrive at \( \|d(f(x, Sx))\| = 0 \). Hence \( f x = Sx \).

Similarly by using the inequality

\[
d(f(x, T x)) \leq d(f(x, f u_{2k+1})) + d(f u_{2k+1}, T x)
\]

We can show that \( f x = T x \), implying that \( y \) is a common point of coincidence of \( S, T \) and \( f \); i.e. \( y = f x = Sx = T x \). Now we show that \( S, T \) and \( f \) have unique point of coincidence. For this, assume that there is another point \( y^* \) in \( X \) such that \( y^* = f x^* = S x^* = T x^* \) for some \( x^* \in X \). Now

\[
d(y, y^*) = d(S x^*, T x^*)
\]

\[
\leq d(f(x^*, S x^*)) \left[ \frac{1+d(f(x, T x))}{1+d(f(x, f x^*))} \right] + b d(f(x, S x)) + d(f(x^*, T x^*)) + c d(f(x, f x^*))
\]

\[
= d(f(x^*, f x^*)) \left[ \frac{1+d(f(x, T x))}{1+d(f(x, f x^*))} \right] + b d(f(x, f x)) + d(f(x^*, f x^*)) + c d(f(x, f x^*))
\]

The last inequality gives

\[
d(y, y^*) \leq c d(y, y^*)
\]

This is possible only when \( y = y^* \).

If \( (S, f) \) and \( (T, f) \) are weakly compatible, then
\[ S_y = Sfx = fSx = fy \quad \text{and} \quad T_y = Tf x = fT x = fy \]

It implies that \( S_y = T_y = fy = z \) (say). Hence, \( z \) is a point of coincidence of \( S, T \) and \( f \) and so \( y = z \) by uniqueness. Thus \( y \) is the unique common fixed point of \( S, T \) and \( f \).

**III. Competing Interests**

Authors have declared that no competing interests exist.

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**References**


