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Certain Representations of Mock Theta Functions

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***Abstract**: In this paper, an attempt has been made to obtain transformation formula for partial mock theta functions. The double series representation for mock-theta functions of order five and seven is obtained in a much-closed form.*

***Keywords**: Mock-Theta function, Double Series, Transformation formula*

I. Introduction

Denis [1], making use of the identity

$$e_q(xt-xyt)e_q(t-tx)=e_q(t-xyt) \tag{1.1}$$

Established the following result:

$$\sum_{m=0}^{n}\binom{n}{m}\frac{x^m[(a)]_m}{[(b)]_m}\Phi\left[\begin{matrix}q^{-n+m},(a)q^m;q^{n-m}\\(b)q^m\end{matrix}\right]$$

$$\Phi\left[\begin{matrix}q^{-m},(c);yq^m\\(d)\end{matrix}\right]=\Phi\left[\begin{matrix}q^{-n},(a)(c);xyq^n\\(b),(d)\end{matrix}\right] \tag{1.2}$$

In this paper, making use of (1.2) an attempt has been made to obtain transformation formula for partial mock theta functions. Transformation formula for mock theta functions of order three established in this paper, had been obtain by N. J. Fine [2] by making use of different transformation formula for the function $F(a,b;t;q)$ which is defined as

$$F(a,b;t;q)=\sum_{n=0}^{\infty}\frac{(aq)_n}{(bq)_n}t^n \tag{1.3}$$

Our aim is to show that one can easily establish these transformation formulas for mock theta functions of order three (earlier established by Fine from a single result (1.2) by specializing the parameters.) We shall also show that with the help of (1.2) double series representation for mock-theta functions of order five and seven can be obtained in a very closed form.

II. Main Results

In this section we shall establish transformation formula for mock-theta functions.

Putting $C=D=0,Y=\beta,B+1,b_{B+1}=q^{-n}$ and xq^{-n} for x in (1.1), we find:

$$\sum_{m=0}^{n}\sum_{t=0}^{n-m}\frac{[(a)]_{m+t}(\beta)_m(-)^m x^{m+t}q^{-mt}}{(q)_m(q)_t[(b)]_{m+t}(q)^{\frac{m(m-1)}{2}}}=\sum_{r=0}^{n}\frac{[(a)]_r(x\beta)^r}{(q)_r[(b)]_r} \tag{2.1}$$

Again taking A+1 for A, $a_{A+1}=\alpha,\frac{x}{\alpha}$ for *x* in (2.1) and $\alpha\to\infty$, gives:

$$\sum_{m=0}^{n}\sum_{t=0}^{n-m}\frac{[(a)]_{m+t}(\beta)_m(-)^t x^{m+t}q^{\frac{t(t-1)}{2}}}{(q)_m(q)_t[(b)]_{m+t}}=\sum_{r=0}^{n}\frac{[(a)]_r(-)^r(x\beta)^r q^{\frac{r(r-1)}{2}}}{(q)_r[(b)]_r} \tag{2.2}$$

Taking $C=D=1,c_1=\alpha,d_1=\beta,\frac{\beta}{\alpha}$ for Y and then $\alpha\to\infty$ in (1.2), we get:

$$\sum_{m=0}^{n}\sum_{t=0}^{n-m}\frac{[(a)]_{m+t}\left(q^{-n}\right)_{m+t}(-)^m x^{m+t} q^{(n-m)t+mn}}{(q)_m (q)_t (\beta)_m [(b)]_{m+t}\, q^{\frac{(m-1)m}{2}}} = \sum_{r=0}^{n}\frac{\left(q^{-n}\right)_r [(a)]_r \left(-x\beta q^n\right)^r q^{\frac{r(r-1)}{2}}}{(q)_r [(b)]_r (\beta)_r} \tag{2.3}$$

Now taking A+1 for A, B+1 for B, $b_{B+1} = q^{-n}$ $a_{A+1} = \alpha, \frac{x}{\alpha} q^{-n}$ for *x* in (2.3), then $\alpha \to \infty$, gives:

$$\sum_{m=0}^{n}\sum_{t=0}^{n-m}\frac{[(a)]_{m+t}(-)^m x^{m+t} q^{\frac{(t-1)t}{2}}}{(q)_m (q)_t (\beta)_m [(b)]_{m+t}} = \sum_{r=0}^{n}\frac{[(a)]_r (x\beta)^r q^{\frac{r(r-1)}{2}}}{(q)_r (\beta)_r [(b)]_r} \tag{2.4}$$

Mock Theta Functions Of Order Three:

Taking *A*=*B*=1, $a_1 = q, b_1 = -q, \beta = -q$ and $x = -1$ in (2.4), we obtain:

$$f_n(q) = \sum_{r=0}^{n}\frac{q^{r^2}}{(-q)_r^2} = \sum_{m=0}^{n}\frac{(-)^m}{(-q)_m^2}\ {}_1\Phi_1\left[\begin{matrix} q^{m+1} ; 1 \\ -q^{m+1} ; q \end{matrix}\right]_{n-m} \tag{2.5}$$

where $f_n(q)$ is a partial mock theta function of order three. Similar notation will be adopted onward for partial mock theta function

For $n \to \infty$ (2.5) yields

$$f(q) = \sum_{r=0}^{\infty}\frac{q^{r^2}}{(-q)_r^2} = \sum_{m=0}^{\infty}\frac{(-)^m}{(-q)_m^2}\ {}_2\Phi_1\left[\begin{matrix} q, 0 \ ; -1 \\ -q \end{matrix}\right] \tag{2.6}$$

Taking A=1,B=1, $a_1 = q, b_1 = iq, \beta = -iq$ and $x = i$ in (2.4), we find:

$$\Phi_n(q) = \sum_{r=0}^{\infty}\frac{q^{r^2}}{\left(-q^2;q^2\right)_r} = \sum_{m=0}^{\infty}\frac{(i)^m}{\left(-q^2;q^2\right)_m}\ {}_1\Phi_1\left[\begin{matrix} q^{m+1} \ ; -i \\ iq^{m+1} \ ; \ q \end{matrix}\right]_{n-m} \tag{2.7}$$

Taking $n \to \infty$ and then summing the inner ${}_1\Phi_1$-series on the right side of (2.7), we get:

$$\Phi(q) = \sum_{r=0}^{\infty}\frac{q^{r^2}}{\left(-q^2;q^2\right)_r} = (1-i)\ {}_2\Phi_1\left[\begin{matrix} q, 0 \ ; -i \\ iq \end{matrix}\right] \tag{2.8}$$

Putting q^2 for *q* then taking A=B=1, $a_1 = q^2, b_1 = q^3, \beta = -q^2$ and *x*=*q* in (2.2), we get:

$$\sum_{r=0}^{n}\frac{q^{r^2+2r}}{\left(q^3;q^2\right)_r} = \sum_{m=0}^{n}\frac{\left(-q^2;q^2\right)_m q^m}{\left(q^3;q^2\right)_m}\ {}_1\Phi_1\left[\begin{matrix} q^{2m+2}; q^2 \ ; -q \\ q^{2m+3} \ ; \ q^2 \end{matrix}\right]_{n-m}$$

which can also be written as

$$\sum_{r=0}^{n}\frac{q^{(r+1)^2}}{\left(q^3;q^2\right)_{r+1}} = \frac{q}{1-q}\sum_{m=0}^{n}\frac{\left(-q^2;q^2\right)_m q^m}{\left(q^3;q^2\right)_m}\ {}_1\Phi_1\left[\begin{matrix} q^{2m+2}; q^2 \ ; -q \\ q^{2m+3} \ ; \ q^2 \end{matrix}\right]_{n-m} \tag{2.9}$$

Now putting r-1 for r in (2.9), we get:

$$\Psi_n(q) = \sum_{r=1}^{n+1}\frac{q^{r^2}}{\left(q;q^2\right)_r} = \frac{q}{1-q}\sum_{r=1}^{n+1}\frac{\left(-q^2;q^2\right)_m q^m}{\left(q^3;q^2\right)_m}\ {}_1\Phi_1\left[\begin{matrix} q^{2m+2}; q^2; -q \\ q^{2m+3}; q^2 \end{matrix}\right]_{n-m}$$

which for taking $n \to \infty$ and then summing the inner ${}_1\Phi_1$-series on the right hand side, yields:

$$\Psi(q) = q\ {}_2\Phi_1\left[\begin{matrix} q^2, -q^2; q^2; q \\ 0 \end{matrix}\right] \tag{2.10}$$

Putting *A*=*B*=1, $a_1 = q, b_1 = \omega^2 q, \beta = -\omega q$ and $x = -\omega^2$ in (2.4), we obtain:

$$x_n(q) = \sum_{r=0}^{n}\frac{q^{r^2}}{\left(1-q-q^2\right)\cdots\left(1-q^r+q^{2r}\right)}$$

$$= \frac{(-)^n \omega^{2m}}{(\omega q_m, -\omega^2 q; q)_m} \; {}_1\Phi_1\begin{bmatrix} q^{m+1} & ; q & ; \omega^2 \\ -\omega^2 q^{m+1} & ; q & \end{bmatrix}_{n-m} \tag{2.11}$$

where $\omega = e^{\frac{2\pi i}{3}}$ and $n \to \infty$, (2.11) yields

$$x_n(q) = \sum_{r=0}^{n} \frac{q^{r^2}}{(1-q-q^2)\cdots(1-q^r+q^{2r})} = (1+\omega^2)\; {}_2\Phi_1\begin{bmatrix} q, & 0 & ; -\omega^2 \\ -\omega q & & \end{bmatrix} \tag{2.12}$$

and taking A=B=1, $a_1 = q, b_1 = q^{\frac{3}{2}}, \beta = q^{\frac{3}{2}}, x = q^{\frac{1}{2}}$ and in (2.4) and then replacing q by q^2 we obtain:

$$\omega_n(q) = \sum_{r=0}^{n} \frac{q^{2r(r+1)}}{(q;q^2)_{r+1}^2} = \frac{1}{(1-q)^2} \sum_{m=0}^{n} \frac{q^m}{(q^3;q^2)_m} \; {}_1\Phi_1\begin{bmatrix} q^{2m+2}; q^2; -q \\ q^{2m+3}; q^2 \end{bmatrix}_{n-m} \tag{2.13}$$

and $n \to \infty$, (2.13) yields

$$\omega(q) = \sum_{r=0}^{\infty} \frac{q^{2r(r+1)}}{(q;q^2)_{r+1}^2} = {}_1\Phi_1\begin{bmatrix} q^{2m+2}; q^2; -q \\ q^{2m+3}; q^2 \end{bmatrix} \tag{2.14}$$

Putting A=1, $a_1 = q,$ and then replacing q by q^2 in (2.2) we get:

$$\sum_{r=0}^{n} \frac{(-)^r (x\beta)^r q^{r(r-1)}}{[(b);q^2]_r} = \sum_{m==0}^{n} \sum_{t=0}^{n-m} \frac{(q^2;q^2)_{m+t} (\beta;q^2)_m (-)^t q^{t(t-1)} x^{m+t}}{(q^2;q^2)_m (q^2;q^2)_t [(b);q^2]_{m+t}} \tag{2.15}$$

Now putting B=1, $b_1 = -q^3, \beta = q$ and $x = q$ in (2.15), we get:

$$\gamma_n(q) = \sum_{r=0}^{n} \frac{q^{r(r+1)}}{[q;q^2]_{r+1}} = \frac{1}{1+q} \sum_{m=0}^{n} \frac{(q;q^2)_m (-)^m q^m x^{m+t}}{(-q^3;q^2)_m} \; {}_1\Phi_1\begin{bmatrix} q^{2m+2}; q^2; -q \\ q^{2m+3}; q^2 \end{bmatrix}_{n-m} \tag{2.16}$$

As $n \to \infty$, (2.16) gives

$$\gamma(q) = \sum_{r=0}^{n} \frac{q^{r(r+1)}}{[-q;q^2]_{r+1}} = {}_2\Phi_1\begin{bmatrix} q^2, & 0 & ; q^2; -q \\ & 0 & \end{bmatrix} \tag{2.17}$$

Taking A=B=1 $a_1 = q, b_1 = \omega^2 q^{\frac{3}{2}}, \beta = \omega q^{\frac{3}{2}}$ and $x = \omega^2 q^{\frac{1}{2}}$ in (2.3) and then replacing q by q^2 we get:

$$\rho_n(q) = \sum_{r=0}^{n} \frac{q^{2r(r+1)}}{(1+q+q^2)\cdots(1+q^{2r+1}+q^{4r+2})}$$

$$= \frac{1}{(1-\omega q)(1-\omega^2 q)} \sum_{m=0}^{n} \frac{(\omega^2 q)^m}{(\omega^2 q^3, \omega q^3; q^2)_m} \times {}_1\Phi_1\begin{bmatrix} q^{2m+2} & ; q^2 & ; \omega^2 q \\ \omega^2 q^{2m+3} & ; q^2 & \end{bmatrix} \tag{2.18}$$

Taking $n \to \infty$ and summing the inner ${}_1\Phi_1$ on the right side of (2.18), we get

$$\rho(q) = \sum_{r=0}^{\infty} \frac{q^{2r(r+1)}}{(1+q+q^2)\cdots(1+q^{2r+1}+q^{4r+2})} = \frac{1}{(1-\omega q)} \; {}_2\Phi_1\begin{bmatrix} q^2, & 0 & ; q^2 & ; \omega^2 q \\ \omega q^3 & & & \end{bmatrix} \tag{2.19}$$

where $\omega = e^{\frac{2\pi i}{3}}$.

Mock Theta Functions Of Order Five:

Putting *A=1*, *B*=0, $a_1 = q, \beta = -q$ and $x = -1$ in (2.4), we find

$$f_{0,n}(q) = \sum_{r=0}^{n} \frac{q^{r^2}}{(-q)_r} = \sum_{m=0}^{n} \sum_{t=0}^{n-m} \binom{m+t}{m} \frac{(-)^m q^{\frac{t(t-1)}{2}}}{(-q)_m}$$

which gives $f_0(q)$ as $n\to\infty$ in the following form

$$f_0(q)=\sum_{r=0}^{\infty}\frac{q^{r^2}}{(-q)_r}=\sum_{m,t=0}^{\infty}\binom{m+t}{m}\frac{(-)^m q^{\frac{t(t-1)}{2}}}{(-q)_m} \tag{2.20}$$

Taking A=4, $a_1=q, a_2=\beta=-q, a_3=iq^{\frac{1}{2}}, a_4=-iq^{\frac{1}{2}}, B=0$ and $x=-1$ in (2.4), we get:

$$\Phi_{0,n}(q)=\sum_{r=0}^{n}\left(-q;q^2\right)_r q^{r^2}=\sum_{m=0}^{n}\sum_{t=0}^{n-m}\binom{m+t;q^2}{m}\left(-q;q^2\right)_{m+t}(-)^m q^{\frac{t(t-1)}{2}}$$

which for $n\to\infty$ gives

$$\Phi(q)=\sum_{r=0}^{n}\left(-q;q^2\right)_r q^{r^2}=\sum_{m=0,t=0}^{\infty}\binom{m+t;q^2}{m}\left(-q;q^2\right)_{m+t}(-q;q)_t q^{\frac{t(t-1)}{2}} \tag{2.21}$$

Taking A=4, $a_1=q, \beta=-q,, B=0$ and $x=-q$ in (2.4), we obtain:

$$f_{1,n}(q)=\sum_{r=0}^{n}\frac{q^{r(r+1)}}{(-q;q)_r}=\sum_{m=0}^{n}\sum_{t=0}^{n-m}\binom{m+t}{m}\frac{(-)^m q^{m+\frac{t(t-1)}{2}}}{(-q)_m}$$ which for $n\to\infty$ gives

$$f_1(q)=\sum_{r=0}^{\infty}\frac{q^{r(r+1)}}{(-q;q)_r}=\sum_{m=0,t=0}^{\infty}\binom{m+t}{m}\frac{(-)^m q^{m+\frac{t(t-1)}{2}}}{(-q)_m} \tag{2.22}$$

Taking B = 0, A= 4, $a_1=q, a_2=\beta=-q, a_3=iq^{\frac{1}{2}}, a_4=-iq^{\frac{1}{2}}$ and $x=-q^2$ in (2.4), we find:

$$\Phi_{1,n}(q)=\sum_{r=0}^{n}\left(-q;q^2\right)_r q^{(r+1)^2}=q\sum_{m=0}^{n}\sum_{t=0}^{n-m}\binom{m+t;q^2}{m}\left(-q;q^2\right)_{m+t}(-q;q)_t(-q)^m q^{2m+\frac{t(t-3)}{2}},$$

which for $n\to\infty$ gives

$$\Phi_1(q)=\sum_{r=0}^{\infty}\left(-q;q^2\right)_r q^{(r+1)^2}=\sum_{m=0,t=0}^{\infty}\binom{m+t;q^2}{m}\left(-q;q^2\right)_{m+t}(-q;q)_t(-q)^m q^{2m+\frac{t(t-3)}{2}} \tag{2.23}$$

Taking *A=2, B=0,* $a_1=q, a_2=-q, x=-1, \beta=q$ in (2.2), we find:

$$\Psi_{1,n}(q)=\sum_{r=0}^{n}\left(-q;q^2\right)_r q^{\frac{r(r+1)}{2}}=\sum_{m=0}^{n}\sum_{t=0}^{n-m}\frac{\left(-q;q^2\right)_{m+t}(-q;q)_t(-q)^m q^{\frac{t(t-1)}{2}}}{(q;q)_t}$$

which, for $n\to\infty$ gives

$$\Psi_1(q)=\sum_{r=0}^{\infty}\left(-q;q^2\right)_r q^{\frac{r(r+1)}{2}}=\sum_{m=0,t=0}^{\infty}\frac{\left(-q;q^2\right)_{m+t}(-q;q)_t(-q)^m q^{\frac{t(t-1)}{2}}}{(q;q)_t} \tag{2.24}$$

Again taking A=2, $a_1=q, a_2=-q, B=0, \beta=q, x=-q$ in (2.4), we find:

$$\Psi_n(q)=\sum_{r=0}^{n}(-q;q)_r q^{\frac{(r+1)(r+2)}{2}}=q\sum_{m=0}^{n}\sum_{t=0}^{n-m}\frac{\left(q^2;q^2\right)_{m+t}(-)^m q^{m+\frac{t(t+1)}{2}}}{(q;q)_t}$$

which, for , $n\to\infty$ yields

$$\Psi(q)=\sum_{r=0}^{\infty}(-q;q)_r q^{\frac{(r+1)(r+2)}{2}}=q\sum_{m=0,t=0}^{\infty}\frac{\left(q^2;q^2\right)_{m+t}(-)^m q^{m+\frac{t(t+1)}{2}}}{(q;q)_t} \tag{2.25}$$

Taking A=1, $a_1=q, B=0, x=\beta=q^{\frac{1}{2}}$ and then replacing q by q^2 in (2.4), we get

$$F_{0,n}(q)=\sum_{r=0}^{n}\frac{q^{2r^2}}{(q;q^2)_r}=\sum_{m=0}^{n}\sum_{t=0}^{n-m}\frac{(q^2;q^2)_{m+t}(-)^m q^{m+t^2}}{(q;q)_{2m}(q^2;q^2)_t}$$ which, for $n\to\infty$ gives

$$F_0(q)=\sum_{r=0}^{\infty}\frac{q^{2r^2}}{(q;q^2)_r}=\sum_{m=0,t=0}^{\infty}\frac{(q^2;q^2)_{m+t}(-)^m q^{m+t^2}}{(q;q)_{2m}(q^2;q^2)_t} \quad (2.26)$$

Taking *A=B=0*, $x=q, y=q^{-n}$, $D+1$ for *D, C+1* for *C*, $d_{D+1}=q^{-n}$, $c_{C+1}=q$ in (1.2), we get

$$\sum_{r=0}^{n}\frac{[(c)]_r}{[(d)]_r}=\sum_{m=0}^{n}\sum_{t=0}^{n-m}\frac{(q^{-m})_t[(c)]_t q^{m+(m-n)t}}{(q)_m[(d)]_t(q^{-n})_t} \quad (2.27)$$

Now taking *C=1*, $c_1=q, D=4, d_1=q^{\frac{1}{2}}, d_2=-q^{\frac{1}{2}}, d_3=q, d_4=-q$ in (2.27), we obtain:

$$\chi_{0,n}(q)=\sum_{r=0}^{n}\frac{q^r}{(q^{r+1};q)_r}=(q)_n\sum_{m=0}^{n}\sum_{t=0}^{m}\frac{(q^{-m})_t(q)_t q^{m+(m-n)t}}{(q)_m(q;q)_{2t}(q^{-n})^t}$$

which, for $n\to\infty$ gives

$$\chi_0(q)=\sum_{r=0}^{\infty}\frac{q^r}{(q^{r+1};q)_r}=(q)_\infty\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\frac{(q^{-m})_t q^{m+mt}(-)^t}{(q)_m(q^{1+t})_{2t}q^{\frac{(t-1)}{2}}} \quad (2.28)$$

Again taking *C*=1, $c_1=q, D=4, d_1=q, d_2=-q, d_3=q^{\frac{3}{2}}, d_4=-q^{\frac{3}{2}}$ in (2.27), we get:

$$\chi_{1,n}(q)=\sum_{r=0}^{n}\frac{q^r}{(q^{r+1};q)_{r+1}}=(q)_n\sum_{m=0}^{n}\sum_{t=0}^{n-m}\frac{(q^{-m})_t q^{m+(1+t)-nt}}{(q)_m(q^{-n})_t(q^{1+t};q)_{t-1}}$$

which, for $n\to\infty$, gives

$$\chi_1(q)=\sum_{r=0}^{\infty}\frac{q^r}{(q^{r+1};q)_{r+1}}=(q)_\infty\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\frac{(q^{-m})_t q^{m+(t+1)}(-)^t}{(q)_m(q^{1+t};q)_{t+1}q^{\frac{t(t-1)}{2}}} \quad (2.29)$$

Mock Theta Function Of Order Seven:

Taking *A*=1, $a_1=q, B=2, b_1=q^{\frac{1}{2}}, b_2=-q^{\frac{1}{2}}, \beta=-q$ and $x=-1$ in (2.4), we obtain:

$$\Im_{o,n}(q)=\sum_{r=0}^{n}\frac{q^{r^2}}{(q^{r+1};q)_r}=\sum_{m=0}^{n}\sum_{t=0}^{n-m}\binom{m+t}{m}\frac{(-)^m q^{\frac{t(t-1)}{2}}}{(-q)_m(q;q^2)_{m+t}}$$

which, for $n\to\infty$, gives

$$\Im_o(q)=\sum_{r=0}^{\infty}\frac{q^{r^2}}{(q^{r+1};q)_r}=\sum_{m=0,t=0}^{\infty}\binom{m+t}{m}\frac{(-)^m q^{\frac{t(t-1)}{2}}}{(-q)_m(q;q^2)_{m+t}} \quad (2.30)$$

Again taking *A*=1, $a_1=q, B=2, b_1=q^{\frac{3}{2}}, b_2=-q^{\frac{3}{2}}, \beta=-q$ and $x=-q^2$ in (2.4), we get:

$$\Im_{1,n}(q)=\sum_{r=0}^{n}\frac{q^{(r+1)^2}}{(q^{r+1};q)_{r+1}}=q\sum_{m=0}^{n}\sum_{t=0}^{n-m}\binom{m+t}{m}\frac{(-)^m q^{2(m+t)}q^{\frac{t(t-1)}{2}}}{(-q)_m(q;q^2)_{m+t+1}}$$

which for $n\to\infty$ gives

$$\Im_1(q)=\sum_{r=0}^{\infty}\frac{q^{(r+1)^2}}{\left(q^{r+1};q\right)_{r+1}}=\sum_{m=0,t=0}^{\infty}\binom{m+t}{m}\frac{(-)^m q^{2m+\frac{(t+1)(t+2)}{2}}}{(-q)_m\left(q;q^2\right)_{m+t+1}} \tag{2.31}$$

Finally taking A=1, $a_1=q, B=2, b_1=q^{\frac{3}{2}}, b_2=-q^{\frac{3}{2}}, \beta=x=-q$ in (2.4), we find:

$$\Im_{2,n}(q)=\sum_{r=0}^{n}\frac{q^{(r+1)}}{\left(q^{r+1};q\right)_{r+1}}=\sum_{m=0}^{n}\sum_{t=0}^{n-m}\binom{m+t}{m}\frac{(-)^m q^{m+\frac{t(t+1)}{2}}}{(-q)_m\left(q;q^2\right)_{m+t+1}}$$

which for $n\to\infty$ gives

$$\Im_2(q)=\sum_{r=0}^{\infty}\frac{q^{(r+1)}}{\left(q^{r+1};q\right)_{r+1}}=\sum_{m=0,t=0}^{\infty}\binom{m+t}{m}\frac{(-)^m q^{m+\frac{t(t+1)}{2}}}{(-q)_m\left(q;q^2\right)_{m+t+1}} \tag{2.32}$$

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