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The existence and uniqueness of Clifford algebra

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Abstract *: Bilinear forms occupy a unique place in all of mathematics .The study of linear transformations alone is incapable of handing the notions of orthogonally in geometry optimization in many variables , Fourier series and so on and so forth . One of the main goals of this work is to explain how we find the existences and uniqueness for Clifford algebra based on bilinear form concept . This is shown by two theorems with their proving.*

Keywords*: bi-linear form, symmetric, quadratic form, Clifford algebra.*

I. Introduction

Geometric algebra was introduced in the nineteen century by the English mathematician William Kingdon Clifford . Clifford (1878) introduced his "geometric algebras " as a generalization of Grassmann algebra , complex numbers and quaternions [1]. Lipschitz (1886) was the first to define groups constructed from "Clifford numbers " and use them to represent rotations in Euclidean space [2]. Cartan E. [3] discovered representations of the Lie algebras $SO_n(\mathbb{C})$ and $SO_n(\mathcal{R})$, n>2 , that do not lift to representations of orthogonal groups . In physics [4] , Clifford algebras and spinors appears for the first time in Pauli's nonrelativistic theory of the "magnetic electron ". Dirac (1928) [5] in his work on the relativistic wave equation of the electron , introduced matrices that provide a representation of the Clifford algebra of Minkowski space . Brauer and Weyl (1935) [6] connected the Clifford and Dirac indices with Cartan's spinorial representations of Lie algebras :they found , in any number of dimensions , the spinorial , projective representations of the orthogonal groups. In [7] Yasuaki Kuroe , Tohru Nitta and Eckhard Hitzer were servey the development of Clifford's geometric algebra and some of its engineering applications during the last 15 years likes : In [9] (the AGACSE 1999 proceeding) has four papers (chp.1 by D.Hestenes , chp.2 by G.Sobezyk , chp.3 by J.M.Pozo and G.Sobczyk , chp.4 by H.Li) introducing the conformal model in Clifford algebra CL(p+1,q+1) . Specialized to CL(n+1,1) , it provides a universal model for conformal geometries of Euclidean , spherical and double- hyperbolic spaces .
In recently papers such as [8] translation , rotation and scale invariant algorithm for registration of color to the original algorithm the proposed algorithm uses the Clifford Fourier transform , also in [8] anew Grassmann algebra (restriction of projective Clifford algebra to the outer product) based framework for global (nD) visibility computations is developed .Jean Gallier [10] in his book define Clifford algebras with their properties as well as introduced Clifford groups and the Pin and Spin groups.

II. Clifford Algebra

Before we are going to give the definition of Clifford algebra in this section , we will define some basic definitions likes : bi-linear and quadratic form in a vector space V over the field F which are very important definitions to help us to explain Clifford algebra . These are shown as follows:

Definition(1):

Let V be a vector space over the field F (predominantly $\mathcal{R}$ or $\mathbb{C}$) . A bi-linear form on a vector space V over a field F is a mapping $\varphi: V \times V \to F$ such that :

1. $\varphi(u+v,w) = \varphi(u,w) + \varphi(v,w)$
2. $\varphi(u,v+w) = \varphi(u,v) + \varphi(u,w)$
3. $\varphi(\lambda u,w) = \lambda\varphi(u,w)$
4. $\varphi(u,\lambda w) = \lambda\varphi(u,w)$

Thus , a bi-linear form on a vector space V is a function on $V \times V$ such that it is linear in both coordinates .

Definition(2):

A bi-linear form $\varphi: V \to F$, then we have φ is symmetric if $\varphi(u,w) = \varphi(w,u) \ \ \forall \ u, w \in V$.

Definition(3):

Let V be an n-dimensional vector space over the field F. Given a symmetric bi-linear form φ on V , the associated quadratic form is the function $\varphi: V \to F$: given by $Q(u) = \varphi(u,u)$.
Notice that Q has the property that : $Q(\lambda u) = \lambda^2 Q(u) \ \forall \ u \in V, \lambda \in F$.

Definition(4):

Let (V,Q) be a vector space with a quadratic form Q . The associated Clifford algebra $CL(V,Q)$ is an associated , unital algebra over the field F with a linear map : $\zeta_Q: V \to CL(V,Q)$ obeying the relation : $\zeta_Q(u)^2 = Q(u).1$, where 1 is the unit element of $CL(V,Q)$.

III. Main results

Theorem(1):

Let V be a vector space with the symmetric bi-linear form $Q(u,v)$ Clifford algebra $CL(V,Q)$. Then there exists Clifford algebra and has dimension 2^n over a field F , where n=dim V .

Proof(1):

Let $\{e_1, . . . , e_n\}$ be the orthonormal basis in V such that : $Q(e_i, e_j) = 0 \ \ \forall \ i \neq j$. We want to show that in $CL(V,Q)$ that :

1. $\zeta(e_i)^2 = a_i$ where $a_i \in F$, $a_i = Q(e_i, e_i)$.
2. $\zeta(e_i)\,\zeta(e_j) = -i\zeta(e_j)\zeta(e_i) \ \ \forall \ i \neq j$.

It follows from the definition of Clifford algebra :

$$\zeta(e_i+e_j)^2 = (\zeta(e_i+e_j))^2 = Q(e_i+e_j, e_i+e_j) = \left(Q(e_i,e_i) + Q(e_i,e_j) + Q(e_j,e_i) + Q(e_j,e_j)\right)$$
$$= a_i + Q(e_i,e_j) + Q(e_i,e_j) + a_j = a_i + a_j + 2Q(e_i,e_j) = a_i + a_j$$

On other hand :

$$(\zeta(e_i+e_j))^2 = \zeta(e_i+e_j)\zeta(e_i+e_j) = (\zeta(e_i)+\zeta(e_j))(\zeta(e_i)+\zeta(e_j)) = \zeta(e_i)^2 + \zeta(e_j)^2 + \zeta(e_i)\zeta(e_j) + \zeta(e_j)\zeta(e_i) = a_i + a_j + \zeta(e_i)\zeta(e_j) + \zeta(e_j)\zeta(e_i)$$

From these equations we get the following equation :

$$a_i + a_j = a_i + a_j + \zeta(e_i)\zeta(e_j) + \zeta(e_j)\zeta(e_i)$$

Therefore,

$$\zeta(e_i)\zeta(e_j) + \zeta(e_j)\zeta(e_i) = 0$$

Now , If Clifford algebra $CL(V,Q)$ exist, then any element of the Clifford algebras $CL(V,Q)$ decomposes on the basis:

$$\{1, \zeta(e_{i_1}), \zeta(e_{i_1})\cdot\zeta(e_{i_2}), \zeta(e_{i_1})\cdot\zeta(e_{i_2})\cdot\zeta(e_{i_3}), \dots, \zeta(e_{i_1})\zeta(e_{i_2})\cdot \dots \cdot\zeta(e_{i_n})\}, \qquad where\ i_1 < \dots < i_n$$

By definition of Clifford algebra : any element of $CL(V,Q)$ is a linear combination of all kinds of works from the elements $\zeta(\mathrm{x})$, where $x \in V$. So as $\{e_i\}$ – an orthonormal basis, then:

$$\zeta(\mathrm{x}) = \zeta(x^i \cdot e_i) = x^i \cdot \zeta(e_i), \zeta(\mathrm{x})\zeta(\mathrm{y}) = x^i y^i \zeta(e_i)\zeta(e_j)$$

Let's take a look at $\zeta(e_{i_1}), \dots, \zeta(e_{i_N})$. If among the $\zeta(e_i)$ meet the same, using ratios, we can rearrange them so that similar stood nearby. We can only change sign, a $\zeta(e_i)^2$ You can replace the a_i and then the number N decrease by 2. In particular, this will always be, If $\mathrm{N} \geq n$, Therefore, it can be assumed that $\mathrm{N} \leq n$. In addition, you can assume that all e_i various. Due to the permutations can be factors we list in ascending order of their numbers, $\zeta(e_{i_1}), \dots, \zeta(e_{i_n}), where\ i_1 < \dots < i_N$.

In order to Show that the Clifford algebra $CL(V,Q)$ exists. This is shown as :

For each subset $S \subset \{1, \dots, n\}$ *Enter symbol* e_s: let $e_\emptyset = 1$($\emptyset$-empty set). Reliable $CL(V,Q)$ linear space K with basis $\{e_s\}$. We define multiplication in $CL(V,Q)$ as follows:

If $1 \leq s,t \leq n$, then : let (s, t) = $\{1\ for\ all\ s \leq t;\ -1\ for\ all\ s > t\}$. For two subsets $S,T \subset \{1,\dots,n\}$, let $a(S,T) = \prod_{s\in S, t\in T}(s,t)\prod_{i\in S\cap T} a_i$ where $a_i = Q(e_i,e_i)$ The product of the linear combinations $\sum a_S e_S$, $\sum b_T e_T \in C(L)$; a_S , $b_T \in K$ define the following formula : $(\sum a_S e_S)(\sum b_T e_T) = \sum a_S b_T\, a(S,T) e_{S\nabla T}$, where $S\nabla T = (S\cup T)\backslash(S\cap T)$ – symmetric difference of sets S, T.

We want to verify that $CL(V,Q)$ is algebra.

1. $e_S\, 0 = 0$
2. $e_S\, e_\emptyset = e_S, e_\emptyset = 1$
3. $Associativity$: $(e_S\, e_T)e_R = e_S\,(e_T e_R)$

$$(e_S\, e_T)e_R = a(S,T)a(S\nabla T,R)e_{(S\nabla T)\nabla R}$$
$$e_S\,(e_T e_R) = a(S,T\nabla R)a(T,R)e_{S\nabla(T\nabla R)}$$
$$e_{(S\nabla T)\nabla R} = e_{S\nabla(T\nabla R)}$$

To do this we will first to show that: $e_S\, e_\emptyset = a(S,\emptyset)e_{S\nabla\emptyset} = e_S$, $a(S,\emptyset) = \prod_{s\in S}(s,\emptyset)\prod_{i\in S\cap\emptyset} a_i$, $e_{S\nabla\emptyset} = e_\emptyset = 1$, $S\nabla\emptyset = (S\cap\emptyset)\backslash(S\cup\emptyset) = \emptyset$: To do this, make sure that $(S\nabla T)\nabla R = S\nabla(T\nabla R)$. To convert expressions use the following formula: $A\nabla B = (A\cup B)\backslash(A\cap B)$, $A\backslash B = A\cap B'$, $(A\cap B)' = A'\cup B'$, $(A\cup B)\cap C = (A\cap B)\cup(B\cap C)$, $A\cap A' = 0$, $0\cup A = A$. Transform $S\nabla T$:
$S\nabla T = (S\cup T)\backslash(S\cap T) = (S\cup T)\cap(S\cap T)' = (S\cup T)\cap(S'\cup T') = [(S\cup T)\cap S']\cup[(S\cup T)\cap T']$=$[(S\cap S')\cup(T\cap S')]\cup[(S\cap T')\cup(T\cap T')] = (T\cap S')\cup(S\cap T')$
Substitute the expression obtained for $S\nabla T$ в $(S\nabla T)\nabla R$.
$(S\nabla T)\nabla R = [(T\cap S')\cup(S\cap T')]\nabla R = \{R\cap[(T\cap S')\cup(S\cap T')]'\}\cup$
$\{[(T\cap S')\cup(S\cap T')]\cap R'\} = \{R\cap[(T\cap S')'\cap(S\cap T')']\}\cup$
$[(R'\cap T\cap S')\cup(R'\cap S\cap T')] = \{R\cap[(T'\cup S)\cap(S'\cup T)]\}\cup$
$[(S'\cap T\cap R')\cup(S\cap T'\cap R')] = \{R\cap[(T'\cup S)\cap S']\cup[(T'\cup S)\cap T]\}\cup$
$[(S'\cap T\cap R')\cup(S\cap T'\cap R')]$=$\{R\cap[(S'\cap T')\cup(S'\cap S)]\cup[(T\cap T')\cup(T\cap S)]\}\cup$
$[(S'\cap T\cap R')\cup(S\cap T'\cap R')]$=$\{R\cap[(S'\cap T')\cup(T\cap S)]\}\cup[(S'\cap T\cap R')\cup(S\cap T'\cap R')] =$
$(S'\cap T'\cap R)\cup(S\cap T\cap R)\cup(S'\cap T\cap R')\cup(S\cap T'\cap R')$.
Similarly convert $S\nabla(T\nabla R)$.
$T\nabla R$=$(T\cup R)\backslash(T\cap R) = (T\cup R)\cap(T\cap R)' = (T\cup R)\cap(T'\cup R') = [(T\cup R)\cap T']\cup[(T\cup R)\cap R'] =$
$[(T'\cap T')\cup(R\cap T')]\cup[(T\cap R')\cup(R\cap R')] = (R\cap T')\cup(T\cap R')$.
$S\nabla(T\nabla R) = S\nabla[(R\cap T')\cup(T\cap R')] = \{S\cap[(R\cap T')\cup(T\cap R')]'\}\cup$
$\{[(R\cap T')\cup(T\cap R')]\cap S'\}$=$\{S\cap[(R\cap T')'\cap(T\cap R')']\}\cup\{S'\cap[(R\cap T')]\cup[S'\cap(T\cap R')]\}$=$\{S\cap$
$[(R'\cup T)\cap(T'\cup R)]\}\cup[(S'\cap T'\cap R)\cup(S'\cap T\cap R')] = \{S\cap[((R'\cup T)\cap T')\cup((R'\cup T)\cap R)]\}\cup$
$[(S'\cap T'\cap R)\cup(S'\cap T\cap R')] = \{S\cap[((R'\cap T')\cup(T\cap T'))\cup((R'\cap R)\cup(T\cap R))]\}\cup(S'\cap T'\cap R)\cup$
$(S'\cap T\cap R') = \{S\cap[(R'\cap T')\cup(T\cap R)]\}\cup(S'\cap T'\cap R)\cup(S'\cap T\cap R') = (S\cap T'\cap R')\cup(S\cap T\cap R)\cup$
$(S'\cap T'\cap R)\cup(S'\cap T\cap R')$.
$Equating\ (S\nabla T)\nabla R\ and\ S\nabla(T\nabla R)$, get match. Therefore, $e_{(S\nabla T)\nabla R} = e_{S\nabla(T\nabla R)}$. In $a(S,T)a(S\nabla T,R)$ the sign is determined by the product of the: $\prod_{\substack{s\in S\\ t\in T}}(s,t)\prod_{\substack{u\in S\nabla T\\ r\in R}}(u,r)$. Make the second work we go first all the elements of S, then all elements of t (for fixed r).
$Let\ us\ introduce\ the\ factors\ (u,r)^2$, where $u\in S\cap T$ equal to 1. So the sign can be written to by S, i. e., piled, R as $\prod_{\substack{s\in S\\ t\in T}}(s,t)\prod_{\substack{u\in S\\ r\in R}}(u,r)\prod_{\substack{u\in T\\ r\in R}}(u,r)$.
Similarly, in $a(S,T\nabla R)a(T,R)$, the sign is determined by the product of

$$\prod_{\substack{s\in S\\ u\in T\nabla R}}(s,u)\prod_{\substack{t\in T\\ r\in R}}(t,r)$$

Make the first work we go first all the elements of t, then all elements of R (with a fixed s).
Let us introduce the factors $(s,u)^2$, where $u\in T\cap R$, equal to 1 . So the sign can be written by S i.e . piled R as:

$$\prod_{\substack{s\in S\\ u\in T}}(s,u)\prod_{\substack{s\in S\\ u\in R}}(s,u)\prod_{\substack{t\in T\\ r\in R}}(t,r)$$

Consider factors that include scalar squares a_i: For $a(S,T)a(S\nabla T,R)$ they have the appearance of

$$\prod_{i\in S\cap T} a_i \prod_{j\in(S\nabla T)\cap R} a_j$$

$so\ as (S\nabla T)\cap R = (S\cap R)\nabla(T\cap R)$, a $S\cap T$ with this many does not intersect, and $(S\cap T)\cup[(S\cap R)\nabla(T\cap R)]$ consists of those elements $S\cup T\cup R$, contained in more than one of these three sets. Therefore, the multiplier symmetrically depends on S, t, R. $For\ a(S,T\nabla R)a(T,R)$. They have the appearance of $\prod_{j\in S\cap(T\nabla R)} a_j\prod_{i\in T\cap R} a_i$. So as $S\cap(T\nabla R) = (S\cap T)\nabla(S\cap R)$, a $T\cap R$ with this multiple does not intersect, and $(T\cap R)\cup[(S\cap T)\nabla(S\cap R)]$ consists of those elements $S\cup T\cup R$, contained in more than one of these three sets. Therefore, the multiplier symmetrically depends on S, t, R. This completes the proof of the associativity of algebra $CL(V,Q)$.
Define F -linear map $\zeta: V\to CL(V,Q)$ condition $\zeta(e_i) = e_{\{i\}}$. Under multiplication formulas , $e_\emptyset = 1$ in $CL(V,Q)$ and $\zeta(e_i)\zeta(e_j) = e_{\{i\}}\, e_{\{j\}} = \{a_i\, e_\emptyset\ for\ all\ i=j;\ -e_{\{j\}}e_{\{i\}}\ for\ all\ i\neq j\}$.
Check out these ratios for the algebra $CL(V,Q)$.
For all $i\neq j$ we have : $e_{\{i\}}\, e_{\{j\}} = a(\{i\},\{j\})e_{\{i\}\nabla\{j\}} = (i,j)e_{\{i,j\}}$.
Let $S=\{i\}, T=\{j\}$, when $a(\{i\},\{j\}) = (i,j)a_\emptyset = (i,j)$;
$\{i\}\nabla\{j\} = \{i,j\}\backslash(\{i\}\cap\{j\}) = \{i,j\}, so\ as\ to\ \{i\}\cap\{j\} = \emptyset$.
For all $i=j$ we have $e_{\{i\}}\, e_{\{i\}} = a(\{i\},\{i\})e_{\{i\}\nabla\{i\}} = a_i$
So as $a(\{i\},\{i\}) = (i,i)a_i = a_i$, $if\ (i,i) = 1$.

Really , $e_{\{i\}\nabla\{i\}} = e_{\emptyset} = 1 \; and \; \{i\}\nabla\{i\} = \{i\}\backslash\{i\} = \emptyset$.
Thus, we have proven that the Clifford algebra $CL(V,Q)$ exists.

Theorem(2):
Let $\sigma: V \to D$ be any F -linear map F –linear map V in F-algebra D , for which $\sigma(u)^2 = Q(u,u).1 \; \forall 1 \in V$. Then there is a unique homomorphism F-algebra $\tau : CL(V,Q) \to D$, such that $\sigma = \tau \circ Q$. In particular , $CL(V,Q)$ is unambiguous with a precision up to isomorphism.
Proof(2):
Let $\sigma: V \to D$ linear map with $\sigma(V)^2 = Q(1,1).1$. There is a unique k-linear map $\tau: CL(V,Q) \to D$, which elements of the basis e_s defined by the formula: $\tau\left(e_{\{i_1\cdots i_m\}}\right) = \sigma\left(e_{i_1}\right)\cdots\sigma\left(e_{i_m}\right), \tau(e_\emptyset) = 1_D$.
For $\tau \cdot \zeta = \sigma$, because it is so on the elements of the basis V. $\tau: CL(V,Q) \to D$ is homomorphism algebra , indeed it is because $\tau(e_S e_T) = \tau(a(S,T)e_{S\nabla T}) = a(S,T)\tau(e_{S\nabla T})$. Check that the $\tau(e_S)\,\tau(e_T)$ we can bring to mind $a(S,T)\tau(e_{S\nabla T})$, taking a advantage of the ratios :$\sigma(e_i)^2 = a_i, \; \sigma(e_i)\sigma\left(e_j\right) = -\sigma\left(e_j\right)\sigma(e_i) \; for \; all \; i \neq j$.
Let $S = \left\{e_{i_1}, \dots, e_{i_m}\right\}, where \; i_1 < \cdots < i_m \; and \; T = \left\{e_{j_1}, \dots, e_{j_1}\right\}, where \; j_1 < \cdots < j_1$.
Then $\tau(e_S)\,\tau(e_T) = \sigma\left(e_{i_1}\right)\dots\sigma\left(e_{i_m}\right)\sigma\left(e_{j_1}\right)\dots\sigma\left(e_{j_1}\right) = \tau(e_S \cdot e_T) = \tau(a(S,T)e_{S\nabla T})$.
Because $e_S e_T = a(S,T)e_{S\nabla T}$, a $\tau\left(e_{\{i_1\cdots i_m\}}\right) = \sigma\left(e_{i_1}\right)\cdots\sigma\left(e_{i_m}\right)$.
Let $\tau(e_S) = e_S^\sigma, \tau(e_T) = e_T^\sigma$ and rewrite the equality in the following form:

$$\tau(e_S)\,\tau(e_T) = e_T^\sigma\, e_S^\sigma = a(S,T)e_{S\nabla T}^\sigma = a(S,T)\tau(e_{S\nabla T})$$

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