Interpolation of Fuzzy if-then rules in context of Zadeh's Z-numbers

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Abstract: Zadeh introduced the concept of Z-numbers in 2011 mathematically to represent the normal everyday verbal reasoning. It is expected to have many applications especially in the field of 'computing with words'. In this paper we develop the basic ideas which will be instrumental to solve the problem of interpolation of fuzzy if-then rules in terms of Zadeh’s Z-numbers.

Keywords: Zadeh’s Z-numbers, fuzzy logic, fuzzy event probability, interpolation, if-then rules, approximate reasoning, computing with words.

I. Introduction

The problem of interpolation arises quite naturally in many applications of fuzzy logic. In fuzzy logic the problem of interpolation is as follows:

Suppose we are given a collection of fuzzy if-then rules of the form:

If $X$ is $A_i$ then ($Y$ is $B_i$), $i = 1,\ldots, n$ where the $A_i$ and $B_i$ are fuzzy sets with specified membership functions.

However if $X$ is $A$, where $A$ is not one of the $A_i$, then the problem of interpolation is to deduce an if-then rule of the form

"If $X$ is $A$ then $Y$ is $B$."

"If $X$ is $A$ then $Y$ is $B$.” using the given collection of rules.

In his paper published in 2011, Zadeh had commented that the problem of interpolation can be generalized to the if-then rules for Z-numbers also. The aim of this paper is to study the interpolation problem in terms of Z-numbers.

A Z-number is an ordered pair of fuzzy numbers $Z = (A, B)$. It is associated with some uncertain variable say $X$. The first component $A$ refers to values which $X$ may take and the second component $B$ refers to reliability or probability that $X$ takes some value in $A$.

For example consider the statement "Usually the train arrives about half an hour late" If $X$ denotes the delay in the arrival time of train, and $A$ is the fuzzy set 'about half an hour' and $B$ is the fuzzy set representing 'usually' then the statement may be represented as $X$ is $z(A, B)$. Here $(A, B)$ is a Z-number. The statement "$X$ is $z(A, B)$.” is called a Z-valuation.

Before describing the interpolation problem in terms of Z-numbers we shall give the necessary formal definitions.

II. Preliminaries

Definition 1: Fuzzy Set

A fuzzy set $A$ defined on a universe $X$ may be given as:

$$A = \left\{ \left( x, \mu_A(x) \right) / x \in X \right\}$$

where $\mu_A: X \rightarrow [0,1]$ is the membership function of $A$. The membership value $\mu_A(x)$ describes the degree of belonging of $x \in X$ in $A$ [2].

Definition 2: Trapezoidal Fuzzy Number

A Trapezoidal Fuzzy Number (TrFN)[3] $A$ is denoted as $(a_1, a_2, a_3, a_4)$ where the membership function
\[
\mu_A(x) = \begin{cases} 
0, & x \leq a_1 \\
\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\
1, & a_2 \leq x \leq a_3 \\
\frac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4 \\
0, & x \geq a_4
\end{cases}
\]

**Definition 3: \(\alpha\)-cuts of a fuzzy sets**

Given a fuzzy set \(A\) on \(U\), a domain and a number \(\alpha\) in \(I\), such that \(0 < \alpha \leq 1\), then associate a crisp set with \(A\), denoted by \(A_\alpha\), defined as \(A_\alpha = \{ x \in U / A(x) \geq \alpha \geq 0 \}\) is called the \(\alpha\)-cuts of \(A\). For each \(\alpha\), we obtain an \(\alpha\)-cut of \(A\). [1]

**Definition 4: Restrictions**

A restriction may be viewed as generalized constraint. Suppose \(A\) is a fuzzy set and \(X\) a variable. The statement \(R(X)\) : \(X\) is \(A\) is referred to as a possibilistic restriction ([4],[5]). Here \(A\) plays the role of possibility distribution of \(X\). The statement "\(R(X) : X\) is \(A\)" is to be understood as" The Possibility (\(X=u\)) is \(\mu_A(u)\) "where \(\mu_A\) is the membership function of \(A\) and \(u\) is a generic value of \(X\).

When \(X\) is a random variable, the probability distribution of \(X\) plays the role of probabilistic restriction on \(X\) ([4],[6]). A probabilistic restriction is expressed as:

\[ R(X): X \text{ is } p, \quad \text{where } p \text{ is the probability density function of } X. \]

That statement "\(R(X): X \text{ is } p\)" is to be understood as “Prob(\(u \leq X \leq u+du\)) = f(u)du”

**Definition 5: Fuzzy Event Probability**

Zadeh [7] has given the following definition for the probability of a fuzzy event:

Let \(X\) be a random variable taking real values and \(A\) a fuzzy set defined on the real line. The fuzzy event probability of “\(X\) is \(A\)” is \(\text{FEP}(x \in A) = \int \mu_A(u) p(u)du\)

**Definition 6: Z-number**

Zadeh [8] defines Z-number as follows:

A Z-number is an ordered pair of fuzzy numbers \(Z = (A,B)\), associated with a real-valued uncertain variable \(X\), with the first component \(A\), a restriction on the values which \(X\) can take and the second component \(B\), a measure of reliability of the first component.

**Note:** Though Zadeh has defined a Z-number to be an ordered pair of fuzzy numbers (A, B), the components A and B may not be fuzzy numbers in the strict technical sense. Rather A should be considered to be a fuzzy subset defined on the real line and B should be considered to be a fuzzy set defined on the interval \([0, 1]\).

**Definition 7: Z-valuation**

A Z-valuation is an ordered triple \((X, A, B)\) where \(A\) and \(B\) are fuzzy numbers [8]. A Z-valuation is equivalent to an assignment statement “\(X\) is \((A, B)\)”, where \(X\) is an uncertain variable. \(A\) is a restriction on the values which \(X\) can take and \(B\) is referred to as certainty ([5],[9]).

Z-valuation \((X, A, B)\) may be viewed as a restriction on \(X\) defined by

\[ \text{FEP}(X \text{ is } A) = B \]

More explicitly

Possibility (\((\text{FEP })(x;A) = u) = \mu_B(u)\)

**Definition 8: Z-restriction**

A Z-restriction is expressed as

\[ R(X): X \text{ is } Z \]

Where \(Z\) is a combination of possibilistic and probabilistic restriction defined as ([5], [4])
Z: FEP(X is A) is B in which A and B are fuzzy numbers
Equivalently, a Z-valuation (X, A, B) is a Z-restriction on X.

\[(X, A, B) \mapsto X \text{ is } (A, B)\]

Definition 9: Standard union operator
Let \(U\) be a domain and \(A, B\) be fuzzy sets on \(U\) [1]. Then, Union of \(A\) and \(B\), denoted by \(A \cup B\), is defined as that fuzzy set on \(U\) for which \((A \cup B)(x) = \max(A(x), B(x))\) for every \(x\) in \(U\).

Definition 10: Standard intersection operator
Let \(U\) be a domain and \(A, B\) be fuzzy sets on \(U\) [1]. Then, Intersection of \(A\) and \(B\), denoted by \(A \cap B\), is defined as that fuzzy set on \(U\) for which \((A \cap B)(x) = \min(A(x), B(x))\) for every \(x\) in \(U\).

III. Interpolation Problem in the Context of Zadeh's Z-numbers
One possible generalization of interpolation in terms of Z numbers is as follows.

Given that

- If \(X\) is \(z(A_1, B_1)\), then \(Y\) is \(z(C_1, D_1)\)
- If \(X\) is \(z(A_2, B_2)\), then \(Y\) is \(z(C_2, D_2)\)

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- If \(X\) is \(z(A_n, B_n)\) then \(Y\) is \(z(C_n, D_n)\)
and also that \(X\) is \(z(A, B)\) (where \(A\) is not one of the \(A_i\)'s.)

The interpolation problem is to deduce an if-then \(z\)-fuzzy rule of the form

If \(X\) is \(z(A, B)\) then \(Y\) is \(z(C, D)\)

For example,

Let \(X\) denotes annual rainfall in a region and \(Y\) denotes crop production in that region.

Given the statements

- If \(X\) is \(z(90\text{ to }110\%\text{ of normal annual rainfall , likely})\) then \(Y\) is \(z(\text{Good harvest , very likely})\)
- If \(X\) is \(z(80\text{ to }120\%\text{ of normal annual rainfall , likely})\) then \(Y\) is \(z(\text{Somewhat good harvest , very likely})\).

Then in a situation where it is likely that rainfall is 95\% to 105\% of normal annual rainfall what can we deduce?

Obviously, we can model this as an interpolation problem involving \(Z\)-fuzzy rules. In this paper we shall study the interpolation problem. It will be difficult to deal with the most general form straight away. Hence we shall start with simple problems at first.

IV. Interpolation of two \(z\)-fuzzy rules
First we shall study problems of the following type:

Given: If \(X\) is \(z(A_1, B)\), then \(Y\) is \(z(C_1, D)\)

\[\text{if } X \text{ is } z(A_2, B) \text{ then } Y \text{ is } z(C_2, D)\]

\[X \text{ is } z(A_3, B)\]

The problem is to calculate \(C_3\) so that we may arrive at the conclusion as

If \(X\) is \(z(A_3, B)\) then \(Y\) is \(z(C_3, D)\).

Case 1: \(A_1 \subset A_2 \subset A_3\) or \(A_1 \subset A_3 \subset A_2\)
Let us first consider the case when \(A_i\)'s and \(C_i\)'s are trapezoidal fuzzy numbers.

Let \(A_1, A_2, A_3, C_1, C_2, C_3\) be denoted by \((\alpha_1, \beta_1, \gamma_1, \delta_1), (\alpha_2, \beta_2, \gamma_2, \delta_2), (\alpha_3, \beta_3, \gamma_3, \delta_3), (l_1, m_1, n_1, p_1)\), \((l_2, m_2, n_2, p_2)\), \((l_3, m_3, n_3, p_3)\) respectively [Figures 1 and 2].

So it seems natural that \(C_2\) must be interpolated between \(C_1\) and \(C_3\) in the same proportion as \(A_2\) is interpolated between \(A_1\) and \(A_3\).
To calculate $C_3$: By using the formulae,

\[
\begin{align*}
\frac{\alpha_1 - \alpha_2}{\alpha_3 - \alpha_2} &= \frac{l_1 - l_2}{l_3 - l_2} \\
\frac{\beta_1 - \beta_2}{\beta_3 - \beta_2} &= \frac{m_1 - m_2}{m_3 - m_2} \\
\frac{\gamma_1 - \gamma_2}{\gamma_3 - \gamma_2} &= \frac{n_1 - n_2}{n_3 - n_2} \\
\frac{\delta_1 - \delta_2}{\delta_3 - \delta_2} &= \frac{p_1 - p_2}{p_3 - p_2}
\end{align*}
\]

$(l_3, m_3, n_3, p_3)$ can be calculated.

In the more general case $C_3$ can be calculated in terms of the alpha cuts.

Given: $\alpha$ – cut of $A_1, A_2, A_3, C_1, C_2$ are $(l_1, m_1), (l_2, m_2), (l_3, m_3), (p_1, q_1), (p_2, q_2)$ respectively.

We can calculate $\alpha$ – cut of $C_3 (p_3, q_3)$ by using the formulae,

\[
\begin{align*}
\frac{l_1 - l_2}{l_2 - l_3} &= \frac{p_1 - p_3}{p_2 - p_3} \\
\frac{m_1 - m_3}{m_2 - m_3} &= \frac{q_1 - q_3}{q_2 - q_3}
\end{align*}
\]

**Case 2**: $A_3 = A_1 \cup A_2$, where $\cup$ is the standard union operator

Obviously here $C_3 = C_1 \cup C_2$

So we have the basic rule

Given: If $X$ is $\mathcal{I}(A_1, B)$, then $Y$ is $\mathcal{I}(C_1, D)$

if $X$ is $\mathcal{I}(A_2, B)$ then $Y$ is $\mathcal{I}(C_2, D)$
Conclusion: If \( X \) is \( z(A_1 \cup A_2, B) \) then \( Y \) is \( z(C_1 \cup C_2, D) \).

Case 3: \( A_3 = A_1 \cap A_2 \) where \( \cap \) is the standard intersection operator.

Obviously here \( C_3 = C_1 \cap C_2 \)

So we have the basic rule

Given: If \( X \) is \( z(A_1, B) \), then \( Y \) is \( z(C_1, D) \)
    
if \( X \) is \( z(A_2, B) \) then \( Y \) is \( z(C_2, D) \)

Conclusion: If \( X \) is \( z(A_1 \cap A_2, B) \) then \( Y \) is \( z(C_1 \cap C_2, D) \).

V. Interpolation of three or more \( z \)-fuzzy rules

The basic rules given above can be combined.

Example: Given: If \( X \) is \( z(A_1, B) \), then \( Y \) is \( z(C_1, D) \)
    
if \( X \) is \( z(A_2, B) \) then \( Y \) is \( z(C_2, D) \)
    
if \( X \) is \( z(A_3, B) \) then \( Y \) is \( z(C_3, D) \)
    
\( X \) is \( z(A_1 \cup (A_2 \cap A_3), B) \).

Then we may conclude: If \( X \) is \( z(A_1 \cup (A_2 \cap A_3), B) \) then \( Y \) is \( z(C_1 \cup (C_2 \cap C_3), D) \).

VI. Conclusion

In this chapter we have studied in detail interpolation of fuzzy if-then \( z \)-rules. So in context where if-then rule corresponding to each basic fuzzy subset is known, using interpolation technique we can deduce the if-then rule for all types of combinations of these basic fuzzy subsets.

References