On Ternary Quadratic Diophantine Equation \( x^2 + 2y^2 = 4z \)

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Abstract: The ternary quadratic diophantine equation \( x^2 + 2y^2 = 4z \) is analyzed for its non-zero distinct integral points on it. A few interesting properties among the solutions are presented.

Keywords: Integral points, Ternary quadratic, Polygonal numbers, Pyramidal numbers and Special numbers.

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Notations:

\( t_{m,n} \) = Polygonal number of rank \( n \) with sides \( m \)

\( ct_{m,n} \) = Centered Polygonal number of rank \( n \) with sides \( m \)

\( P_n \) = Pronic number

\( G_n \) = Gnomonic number

\( S_n \) = Star number

\( CH_n \) = Centered hexagonal number

I. INTRODUCTION

Diophantine equations is an interesting concept, as it can be seen from [1-2]. For an extensive review of various problems one may refer [3-14]. This communication concerns with yet another interesting ternary quadratic diophantine equation \( x^2 + 2y^2 = 4z \) for determining its infinitely many non-zero integral solutions. Also a few interesting properties among the solutions are presented.

II. METHOD OF ANALYSIS

The ternary quadratic equation to be solved for its non-zero integral solution is

\[ x^2 + 2y^2 = 4z \] (1)

To start with, it is seen that (1) is satisfied by the following triples:

\( (\pm 2k, \pm 2k, 3k^2) \) and, \( (10k, \pm 2k, 27k^2) \)

However, we have other patterns of solutions which are illustrated as follows.

PATTERN- I

Introducing the linear transformations

\[ x = X + 2T \], \( y = X - T \), \( z = 3u^2 \] (2)

in (1), it is written as

\[ X^2 + 2T^2 = 4u^2 \] (3)

The general solution of the above equation is given by

\[ \begin{align*}
T &= 2pq \\
X &= 2p^2 - q^2 \\
2u &= 2p^2 + q^2
\end{align*} \] (4)
Since our aim is to find integral solution, substitute q as 2Q in (4).

The value of X, T, and u are

\[
\begin{align*}
X &= 2p^2 - 4Q^2 \\
T &= 4pQ \\
u &= p^2 + 2Q^2
\end{align*}
\]

Using (5) in (2), we get the corresponding non-zero distinct integer solutions of (1) are found to be

\[
x = x(p, Q) = 2p^2 - 4Q^2 + 8pQ \\
y = y(p, Q) = 2(p^2 - 2Q^2 - 2pQ) \\
z = z(p, Q) = 3(p^2 + 2Q^2)^2
\]

**PROPERTIES**

1) \(x(p, p + 1) - y(p, p + 1) = 24t_{5, p}\).
2) \(2z(p, Q)\) is a nasty number.
3) \(12 - [x(p, p - 1) + 2y(p, p - 1)]\) is a nasty number.
4) \(x(1, -1) + y(1, -1) + z(1, -1) \equiv 0\) (mod 13).
5) \(x(1, Q) + y(1, Q) + 4t_{6, Q} \equiv 0\) (mod 2^2).
6) \(x(1, Q) - y(1, Q) - 6G_Q \equiv 0\) (mod 6).
7) \(z(p, 1) - 144DF_p - t_{28, p} - 2P_p - 5G_p \equiv 0\) (mod 17).
8) \(4P_p - x(p, 1) - y(p, 1) \equiv 0\) (mod 2^3).
9) \(x(p - 1, p) - CH_p + 3G_p \equiv 0\) (mod 2).
10) \(2P_{p-1} - G_p) - y(p - 1, 1) \equiv 0\) (mod 6).

**Remark:**

Instead of (2) one may consider the following transformation

\(x = X + 2T, y = X - T, z = 9(4k^2 + 8k + 12)t^2\)

Following the procedure presented in Pattern I, the other choices of solutions to (1) are obtained.

**PATTERN- II**

Assume that

\(z(a, b) = a^2 + 2b^2\) \hspace{1cm} (6)

where \(a, b\) are non zero distinct integers.

Write 4 as

\(4 = \frac{(2 + i4\sqrt{2})(2 - i4\sqrt{2})}{9}\) \hspace{1cm} (7)

Substitute (6) and (7) in (1), we get

\(x^2 + 2y^2 = \frac{(2 + i4\sqrt{2})(2 - i4\sqrt{2})}{9}(a^2 + 2b^2)\)

By the method of factorization, the above equation is written as

\[(x + i\sqrt{2}y)(x - i\sqrt{2}y) = \frac{(2 + i4\sqrt{2})(2 - i4\sqrt{2})}{9}\left(a + i\sqrt{2}b\right)\left(a - i\sqrt{2}b\right)\]

On comparing the positive and negative factors, we get

\((x + i\sqrt{2}y) = \frac{(2 + i4\sqrt{2})}{3}\left(a + i\sqrt{2}b\right)\) \hspace{1cm} (8)
\[(x-i\sqrt{2}y) = \frac{(2-i4\sqrt{2})}{3}(a-i\sqrt{2}b)\] (9)

On comparing real and imaginary parts either in (8) or (9), we have
\[
x = \frac{1}{3}[2a-8b]
\]
\[
y = \frac{1}{3}[4a+2b]
\]
(10)

Since our aim is to be find an integral solutions, Substituting \(b = m, a = 3k + m - 3\) in (6) and (10)

The corresponding non-zero distinct integer solution of (1) are found to be
\[
x = x(k, m) = 2(k - m - 1)
\]
\[
y = y(k, m) = 4k + 2m - 4
\]
\[
z = z(k, m) = (3k + m - 3)^2 + 2m^2
\]

**PROPERTIES**

1) \(x(1, -1) + y(1, m) + z(1, m) = 3t_{4,m}\).
2) \(2x(k, m) - y(k, m) + z(1, m) + 3m = 3p_{m-1}\).
3) \(x(k, k) + y(k, k) + z(k, k) - 6p_{k-1} \equiv 0 \pmod{3}\).
4) \(x(k, k - 1) + y(k, k - 1) - G_k + 5 \equiv 0 \pmod{4}\).
5) \(6[z(m, m) - 4S_m - 5]\) is a nasty number.
6) \(x(k, 1)y(k, 1) - t_{18,k} + 8G_k + k \equiv 0 \pmod{10}\).
7) \(CH_k + t_{1,4,k} - x(k^2, 1) - y(k^2, 1) - z(1, k - 1) \equiv 0 \pmod{2}\).
8) \(z(k - 1, 1) - 2ct_{4,k} + 19G_k - k \equiv 0 \pmod{6}\).
9) \(y(1, m + 1) + z(1, m + 1) - 2(5t_{3,m} - G_m) - m \equiv 0 \pmod{5}\).
10) \(2P_{k-1} - G_k - x(k - 1^2, 1) \equiv 0 \pmod{3}\).

**PATTERN-III**

Equation (1) can be written as
\[x^2 + 2y^2 = 4z \ast 1\] (11)

Write 1 as
\[
1 = \left(1 + i2\sqrt{2}\right)\left(1 - i2\sqrt{2}\right)
\]
(12)

Using (7) and (12) in equation (11), we get
\[x^2 + 2y^2 = \frac{(2 + i4\sqrt{2})}{9}\left[\left(1 + i2\sqrt{2}\right)\left(1 - i2\sqrt{2}\right)\right] (a^2 + 2b^2)\]

By method of factorization,
\[\left(x + i\sqrt{2}y\right)\left(x - i\sqrt{2}y\right) = \frac{(2 + i4\sqrt{2})}{9}\left[\left(1 + i2\sqrt{2}\right)\left(1 - i2\sqrt{2}\right)\right] (a + i\sqrt{2}b)(a - i\sqrt{2}b)\]

On comparing the positive and negative factors, we get
\[\left(x + i\sqrt{2}y\right) = \frac{(2 + i4\sqrt{2})}{3}\left(1 + i2\sqrt{2}\right) (a + i\sqrt{2}b)\] (13)
\[\left(x - i\sqrt{2}y\right) = \frac{(2 - i4\sqrt{2})}{3}\left(1 - i2\sqrt{2}\right) (a - i\sqrt{2}b)\] (14)

Consider the equation (13)
\[
(x + i\sqrt{2}y) = \frac{(2 + i4\sqrt{2})}{3} \left(1 + i2\sqrt{2}\right)\left(a + i\sqrt{2}b\right)
\]

On comparing the real and imaginary parts from the above equation, we get
\[
x = \frac{-2}{9} [7a + 8b]
\]
\[
y = \frac{2}{9} [4a - 7b]
\]
(15)

Since our aim is to find integral solutions, substitute \( b = m, a = 9k + 4m - 9 \)
The corresponding non-zero distinct integer solutions of equation (1) are found to be
\[
x = x(k, m) = -2[7k + 4m - 7]
\]
\[
y = y(k, m) = -2[4k + m - 4]
\]
\[
z = z(k, m) = (9k + 4m - 9)^2 + 2m^2
\]

**PROPERTIES**

1) \(2(x(-1, m) - y(-1, 4m))\) is nasty number.
2) \(3z(1, m)\) is nasty number.
3) \(x(k, k) + z(1, k) + G_k - 18P_{k-1} \equiv 0 \) (mod 13).
4) \(z (k, 1) + x(k1) - 2{(t_{83,k} + ct_{25,k})} \equiv 0 \) (mod 25).
5) \(x(-2k, 1) + y(-2k, 1) + 2{(t_{83,k} - G_k) - S_k} \equiv 0 \) (mod 13).
6) \(x(k, 1) y(k, 1) + z(k, 1) - 8t_{29,k} - 6G_k - 5P_{k-1} + 2t_{19,k} \equiv 0 \) (mod 86).
7) \(z(k + 1, 1) = -8t + 94G_k - 18t_{11,k} + k \equiv 0 \) (mod 104).
8) \(x(k^2, k) = -y(k^2, k) - 6G_k + CH_k = 1\).
9) \(y(k - 1, k^2) + 2ct_{2,k} + 3G_k \equiv 0 \) (mod 7).

**GENERATION OF SOLUTIONS:**
Let \((x_0, y_0, z_0)\) be any given non-zero distinct points on the given equation. It is observed that each of the following triple of integers also satisfies (1).

**Triple 1:** \((x_n, y_n, z_n)\)

where
\[
x_n = 3^{n-1} \left((1 + 2(-1)^n)x + (2 - 2(-1)^n)y_0\right)
\]
\[
y_n = 3^{n-1} \left((1 - (-1)^n)x_0 + (2 + (-1)^n)y_0\right)
\]
\[
z_n = 9^n z_0
\]

**Triple 2:** \((x_n, y_n, z_n)\)

where
\[
x_n = x_0
\]
\[
y_n = y_0 + 2n
\]
\[
z_n = 2ny_0 + z_0 + 2n^2
\]

**Triple 3:** \((x_n, y_n, z_n)\)

where
\[
x_n = x_0 + 4n
\]
\[
y_n = y_0
\]
\[
z_n = 2nx_0 + z_0 + 4n^2
\]
III. CONCLUSION

It is worth to mention that instead of (12) one may consider the following representation for 1

\[ 1 = \left(1 + i \sqrt{2}\right) \left(1 - i \sqrt{2}\right) \]

To conclude one may research for other patterns of solutions and their corresponding properties.

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