New Classes Containing Combination of Ruscheweyh Derivative and a New Generalized Multiplier Differential Operator

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Abstract: New classes containing the linear operator obtained as a linear combination of Ruscheweyh derivative and a new generalized multiplier differential operator have been introduced. Sharp results concerning coefficients and distortion theorems of functions belonging to these classes are determined.

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I. Introduction

Denote by $U$ the open unit disc of the complex plane, $U = \{ z \in \mathbb{C} : |z| < 1 \}$. Let $H(U)$ be the space of holomorphic functions in $U$. Let $A$ denote the family of functions in $H(U)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (1.1)$$

The author has recently introduced the following new generalized multiplier differential operator in [15].

Definition 1.1 Let $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then for $f \in A$, a new generalized multiplier operator $I_{\alpha, \beta}^m$ was defined by

$$I_{\alpha, \beta}^0 f(z) = f(z), \quad I_{\alpha, \beta}^1 f(z) = \frac{\alpha f(z) + \beta zf'(z)}{\alpha + \beta}, \quad \ldots, \quad I_{\alpha, \beta}^m f(z) = I_{\alpha, \beta}^{m-1} (I_{\alpha, \beta}^1 f(z)).$$

Remark 1.2 Observe that for $f(z)$ given by (1.1), we have

$$I_{\alpha, \beta}^m f(z) = z + \sum_{k=2}^{\infty} A_k (\alpha, \beta, m) a_k z^k, \quad (1.2)$$

where

$$A_k (\alpha, \beta, m) = \left( \frac{\alpha + k \beta}{\alpha + \beta} \right)^m. \quad (1.3)$$

We note that: i) $I_{1-\beta, \beta, 0}^m f(z) = D^m f(z), \; \beta \geq 0$ (See F. M. Al-Oboudi [1]), ii) $I_{l+1-\beta, \beta, 0}^m f(z) = I_{l, \beta}^m f(z), \; l > -1, \; \beta \geq 0$ (See A. Catas [3] and he has considered for $l \geq 0$) and iii) $I_{\alpha, 1}^m f(z) = I_{\alpha}^m f(z), \; \alpha > -1$ (See Cho and Srivastava [4]) and Cho and Kim [5]). $D^m f(z)$ was introduced by Salagean [9] and was considered for $m \geq 0$ in [2].

Definition 1.3 (8) For $m \in \mathbb{N}_0, f \in A$, the operator $R^m$ is defined by $R^m : A \rightarrow A$, $R^0 f(z) = f(z), \; R^1 f(z) = zf'(z), \ldots, \; (m+1)R^{m+1} f(z) = z(R^m f(z))' + mR^m f(z), \; z \in U$.

Remark 1.4 If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$, then $R^m f(z) = z + \sum_{k=2}^{\infty} B_k (m) a_k z^k, \; z \in U$, where
\[ B_k(m) = \frac{(m+k-1)!}{m!(k-1)!}. \] (1.4)

The author has introduced the following operator in [16].

**Definition 1.5** Let \( f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \beta \geq 0, \alpha \) a real number such that \( \alpha + \beta > 0 \).

Denote by \( R^m_{\alpha, \beta, \delta} \), the operator given by \( R^m_{\alpha, \beta, \delta} : A \to A, \)

\[ R^m_{\alpha, \beta, \delta} f(z) = (1-\delta)R^m f(z) + \delta A_k(\alpha, \beta, m)z^k, \ z \in U. \]

The operator was studied also in [11], [12], [13] and [14]. Clearly \( R^m_{\alpha, \beta, 0} = R^m \) and \( R^m_{\alpha, \beta, 1} = I_{\alpha, \beta}. \)

**Remark 1.6** If \( f(z) = z + \sum_{k=0}^{\infty} a_k z^k \), then from (1.2) and Remark 1.4, we have

\[ R^m_{\alpha, \beta, \delta} f(z) = z + \sum_{k=2}^{\infty} \left\{ (1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m) \right\} a_k z^k, \ z \in U, \]

where \( A_k(\alpha, \beta, m) \) and \( B_k(m) \) are as defined in (1.3) and (1.4), respectively.

We introduce new classes as below by making use of the generalized operator \( R^m_{\alpha, \beta, \delta} \).

**Definition 1.7** Let \( f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \rho \in [0,1], \sigma \in [0,1], \beta \geq 0, \alpha \) a real number such that \( \alpha + \beta > 0 \).

Then \( f(z) \) is in the class \( \mathbb{T}^m_{\alpha, \beta, \delta}(\sigma, \rho) \) if and only if

\[ \frac{\left| \frac{z^2[R_{\alpha, \beta, \delta}^m f(z)]}{z[R_{\alpha, \beta, \delta}^m f(z)]} \right|^2}{\left| \frac{z[R_{\alpha, \beta, \delta}^m f(z)]}{z[R_{\alpha, \beta, \delta}^m f(z)]} \right|^2 + 1 - 2\rho} < \sigma, \ z \in U. \] (1.5)

**Definition 1.8** Let \( f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \rho \in [0,1], \sigma \in [0,1], \beta \geq 0, \alpha \) a real number such that \( \alpha + \beta > 0 \).

Then \( f(z) \) is in the class \( \mathbb{E}^m_{\alpha, \beta, \delta}(\sigma, \rho) \) if and only if

\[ \frac{\frac{z [R_{\alpha, \beta, \delta}^m f(z)]}{z (R_{\alpha, \beta, \delta}^m f(z))} - 1}{\left| \frac{z [R_{\alpha, \beta, \delta}^m f(z)]}{z (R_{\alpha, \beta, \delta}^m f(z))} \right|^2 + 1 - 2\rho} < \sigma, \ z \in U. \] (1.6)

**Definition 1.9** Let \( f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \rho \in [0,1], \sigma \in [0,1], \beta \geq 0, \alpha \) a real number such that \( \alpha + \beta > 0 \).

Then \( f(z) \) is in the class \( \mathbb{R}^m_{\alpha, \beta, \delta}(\sigma, \rho) \) if and only if

\[ \frac{\frac{z (R_{\alpha, \beta, \delta}^m f(z))}{z (R_{\alpha, \beta, \delta}^m f(z))} - 1}{\left| \frac{z (R_{\alpha, \beta, \delta}^m f(z))}{z (R_{\alpha, \beta, \delta}^m f(z))} \right|^2 + 1 - 2\rho} < \sigma, \ z \in U. \] (1.7)

Let \( T \) denote the subclass of \( A \) consisting of functions whose non-zero coefficients, on second on, are negative; that is, an analytic function \( f \) is in \( T \) if and only if it can be expressed as \( f(z) = z - \sum_{k=2}^{\infty} a_k z^k \), \( a_k \geq 0, \ z \in U \). If \( f \in T \), then \( R^m_{\alpha, \beta, \delta} f(z) = z - \sum_{k=2}^{\infty} \{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k z^k \), where \( A_k(\alpha, \beta, m) \) and \( B_k(m) \) are as defined in (1.3) and (1.4), respectively. We denote by \( T\mathbb{T}_{\alpha, \beta, \delta}(\sigma, \rho), T\mathbb{E}_{\alpha, \beta, \delta}(\sigma, \rho) \) and \( T\mathbb{R}_{\alpha, \beta, \delta}(\sigma, \rho) \), the classes of functions \( f(z) \in T \) satisfying (1.5), (1.6) and (1.7) respectively.

In this paper, sharp results concerning coefficients and distortion theorems for the classes \( T\mathbb{T}_{\alpha, \beta, \delta}(\sigma, \rho), T\mathbb{E}_{\alpha, \beta, \delta}(\sigma, \rho) \) and \( T\mathbb{R}_{\alpha, \beta, \delta}(\sigma, \rho) \) are determined. Throughout this paper, unless otherwise mentioned we shall assume that \( A_k(\alpha, \beta, m) \) and \( B_k(m) \) are as defined in (1.3) and (1.4) respectively.
II. Coefficient bounds.

In this section we study the characterization properties following the papers of V. P. Gupta and P. K. Jain [6, 7] and H. Silverman [10].

**Theorem 2.1** A function $f$ is in $T \mathcal{S}^m_{\alpha, \beta, \delta}(\sigma, \rho)$ if and only if

$$\sum_{k=2}^{\infty} \left\{ k(k+1)(1+\sigma) + 2\sigma(1-2\rho) - 2 \right\} \left\{ (1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m) \right\} a_k^2 \leq 4\sigma(1-\rho). \quad (2.1)$$

The result is sharp.

**Proof.** Suppose $f$ satisfies (2.1). Then for $|z| < 1$, we have

$$\left| z^2 (Rl_{\alpha, \beta, \delta}^m f(z)) - 2RI_{\alpha, \beta, \delta}^m f(z) \right| - \sigma \left| z^2 (RI_{\alpha, \beta, \delta}^m f(z)) \right| + 2(1-2\rho)RI_{\alpha, \beta, \delta}^m f(z) =$$

$$\left| \sum_{k=2}^{\infty} (k+1)(1+\sigma)(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m) a_k z^k \right| -$$

$$\sigma \left| 4(1-\rho) - \sum_{k=2}^{\infty} (k+1)(1+\sigma)(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m) a_k z^k \right| \leq$$

$$\sum_{k=2}^{\infty} (k+1)(1+\sigma)(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m) a_k z^k -$$

$$\sum_{k=2}^{\infty} \sigma(k+1)(1+\sigma)(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m) a_k z^k =$$

$$\sum_{k=2}^{\infty} (k+1)(1+\sigma) + 2\sigma(1-2\rho) - 2 \left\{ (1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m) \right\} a_k - 4\sigma(1-\rho) < 0.$$  

Hence, by using the maximum modulus theorem and (1.5), $f \in T \mathcal{S}^m_{\alpha, \beta, \delta}(\sigma, \rho)$.

For the converse, assume that

$$\left| \frac{\sum_{k=2}^{\infty} (k+1)(1+\sigma)(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m) a_k z^k}{4\sigma(1-\rho) - \sum_{k=2}^{\infty} \sigma(k+1)(1+\sigma)(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m) a_k z^k} \right| < \sigma, \ z \in U.$$  

Since $\Re(z) \leq |z|$ for all $z \in U$, we obtain

$$\Re \left( \frac{\sum_{k=2}^{\infty} (k+1)(1+\sigma)(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m) a_k z^k}{4\sigma(1-\rho) - \sum_{k=2}^{\infty} \sigma(k+1)(1+\sigma)(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m) a_k z^k} \right) < \sigma. \quad (2.2)$$

Choose values of $z$ on the real axis so that $\left( \frac{z^2 (RI_{\alpha, \beta, \delta}^m f(z))}{2RI_{\alpha, \beta, \delta}^m f(z)} \right)$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1$ through real values, we have the desired inequality (2.1).

The function $f_1(z) = z - \frac{4\sigma(1-\rho)}{\left\{ (k+1)(1+\sigma) + 2\sigma(1-2\rho) - 2 \right\} \left\{ (1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m) \right\} z^k}$, $k \geq 2$, is an extremal function for the theorem.

**Theorem 2.2** A function $f$ is in $T \mathcal{L}^m_{\alpha, \beta, \delta}(\sigma, \rho)$ if and only if

$$\sum_{k=2}^{\infty} (2k^2(1+\sigma) + (k+1)(\sigma(1-2\rho) - 1)) \left\{ (1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m) \right\} a_k \leq 4\sigma(1-\rho). \quad (2.3)$$

The result is sharp.

**Theorem 2.3** A function $f$ is in $T \mathcal{R}^m_{\alpha, \beta, \delta}(\sigma, \rho)$ if and only if

$$\sum_{k=2}^{\infty} (k^2(1+\sigma) + \sigma(1-2\rho) - 1) \left\{ (1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m) \right\} a_k \leq 2\sigma(1-\rho). \quad (2.4)$$

The result is sharp.
The proofs of Theorem 2.2 and Theorem 2.3 are similar to that of Theorem 2.1 and so omitted. Extremal functions are given by

\[ f_z(z) = z - \frac{4\sigma(1 - \rho)}{\{2k^2(1 + \sigma) + (k + 1)(\sigma(1 - 2\rho) - 1]\{1 - \delta\}B_k(m) + \delta A_k(\alpha, \beta, m)}} z^k, k \geq 2 \]

and

\[ f_\beta(z) = z - \frac{2\sigma(1 - \rho)}{\{(k^2 + 1 + \sigma)(1 + 2\rho) - 1\}B_k(m) + \delta A_k(\alpha, \beta, m)}} , k \geq 2, \]

respectively.

**Corollary 2.4**

i) If \( f \in T\Sigma^m_{\alpha, \beta, \delta} (\sigma, \rho) \)

then \( a_k \leq \frac{4\sigma(1 - \rho)}{\{k(k + 1)(1 + \sigma) + 2\sigma(1 - 2\rho) - 2\}B_k(m) + \delta A_k(\alpha, \beta, m)}} , k \geq 2, \)

with equality only for the functions of the form \( f_\alpha(z) \).

ii) If \( f \in T\ell^m_{\alpha, \beta, \delta} (\sigma, \rho) \)

then \( a_k \leq \frac{4(1 - \rho)}{(2k^2(1 + \sigma) + (k + 1)(\sigma(1 - 2\rho) - 1))B_k(m) + \delta A_k(\alpha, \beta, m)}} , k \geq 2, \)

with equality only for the functions of the form \( f_\beta(z) \).

iii) If \( f \in T\Sigma^m_{\alpha, \beta, \delta} (\sigma, \rho) \)

then \( a_k \leq \frac{2\sigma(1 - \rho)}{\{(k^2 + 1 + \sigma)(1 + 2\rho) - 1\}B_k(m) + \delta A_k(\alpha, \beta, m)}} , k \geq 2, \)

with equality only for the functions of the form \( f_\beta(z) \).

**III. Distortion theorems**

**Theorem 3.1** If a function \( f(z) \in T \) is in \( T\Sigma^m_{\alpha, \beta, \delta} (\sigma, \rho) \) then

\[ |f(z)| \geq |z| - \frac{\sigma(1 - \rho)}{(1 + \sigma(2 - \rho))(m + 1)(1 - \delta) + \delta A_2(\alpha, \beta, m))} |z|^2, z \in U \]

and

\[ |f(z)| \leq |z| + \frac{\sigma(1 - \rho)}{(1 + \sigma(2 - \rho))(m + 1)(1 - \delta) + \delta A_2(\alpha, \beta, m))} |z|^2, z \in U. \]

Proof In view of Theorem 2.1, we have

\[ 4(1 + \sigma(2 - \rho))(m + 1)(1 - \delta) + \delta A_2(\alpha, \beta, m)) \sum_{k=2}^{\infty} a_k \leq \]

\[ \sum_{k=2}^{\infty} \{k(k + 1)(1 + \sigma) + 2\sigma(1 - 2\rho) - 2\} \{1 - \delta\}B_k(m) + \delta A_k(\alpha, \beta, m) \} a_k \leq 4\sigma(1 - \rho). \]

Thus \( \sum_{k=2}^{\infty} a_k \leq \frac{\sigma(1 - \rho)}{(1 + \sigma(2 - \rho))(m + 1)(1 - \delta) + \delta A_2(\alpha, \beta, m))} . \) So we get

\[ |f(z)| \leq |z| + |z|^2 \sum_{k=2}^{\infty} a_k \leq |z| + \frac{\sigma(1 - \rho)}{(1 + \sigma(2 - \rho))(m + 1)(1 - \delta) + \delta A_2(\alpha, \beta, m))} |z|^2. \]
On the other hand

\[ |f(z)| \geq |z| - |z|^2 \sum_{k=2}^{\infty} a_k \geq |z| - \frac{\sigma(1 - \rho)}{(1 + \sigma(2 - \rho))(m + 1)(1 - \delta) + \delta A_2(\alpha, \beta, m)} |z|^2. \]

**Theorem 3.2** If a function \( f(z) \in T \) is in \( T^m_{\alpha, \beta, \delta} (\sigma, \rho) \) then

\[ |f(z)| \geq |z| - \frac{4\sigma(1 - \rho)}{(5 + \sigma(11 - 6\rho))(m + 1)(1 - \delta) + \delta A_2(\alpha, \beta, m)} |z|^2, z \in U \]

and

\[ |f(z)| \leq |z| + \frac{4\sigma(1 - \rho)}{(5 + \sigma(11 - 6\rho))(m + 1)(1 - \delta) + \delta A_2(\alpha, \beta, m)} |z|^2, z \in U. \]

**Theorem 3.3** If a function \( f(z) \in T \) is in \( T^m_{\alpha, \beta, \delta} (\sigma, \rho) \) then

\[ |f(z)| \geq |z| - \frac{2\sigma(1 - \rho)}{(3 + \sigma(5 - 2\rho))(m + 1)(1 - \delta) + \delta A_2(\alpha, \beta, m)} |z|^2, z \in U \]

and

\[ |f(z)| \leq |z| + \frac{2\sigma(1 - \rho)}{(3 + \sigma(5 - 2\rho))(m + 1)(1 - \delta) + \delta A_2(\alpha, \beta, m)} |z|^2, z \in U. \]

The proofs of Theorem 3.2 and Theorem 3.3 are similar to that of Theorem 3.1.

**Remark 3.4** The bounds of Theorem 3.2 and Theorem 3.3 are sharp since the equalities are attained for the functions

\[ f(z) = z - \frac{4\sigma(1 - \rho)}{(5 + \sigma(11 - 6\rho))(m + 1)(1 - \delta) + \delta A_2(\alpha, \beta, m)} z^2 \quad (z = \pm r) \]

and

\[ f(z) = z - \frac{2\sigma(1 - \rho)}{(3 + \sigma(5 - 2\rho))(m + 1)(1 - \delta) + \delta A_2(\alpha, \beta, m)} z^2 \quad (z = \pm r), \]

respectively.

References


