Conical representation of Rational Quartic Trigonometric Bèzier curve with two shape parameters
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Abstract: A rational quartic trigonometric Bèzier curve with two shape parameters, which is analogous to
quadratic Bèzier curve, is presented in this paper. The shape of the curve can be adjusted as desired, by
simply altering the value of shape parameters, without changing the control polygon. The rational quartic
trigonometric Bèzier curve can be made close to the rational quadratic Bèzier curve or closer to the given
control polygon than the rational quadratic Bèzier curve. The ellipses and circles can be representation
exactly by using rational quartic trigonometric Bèzier curve.

Keywords: Rational Trigonometric Bèzier Basis Function, Rational Trigonometric Bèzier Curve, Shape
Parameter, Open curves, Close curves.

I. Introduction
Rational spline is a commonly used spline function. In many cases the rational spline curves better
approximating functions than the usual spline functions. It has been observed that many simple shapes including
conic section and quadric surfaces can not be represented exactly by piecewise polynomials, whereas rational
polynomials can exactly represent all conic sections and quadric surfaces in an easy manner (see Sarfraz, M.
And Habib, Z. [11]). Many authors have studied this spline, especially the rational cubic spline see; Duan, Q.,
Recently, many papers investigate the trigonometric Bèzier-like polynomial, trigonometric spline and their
applications. Many kinds of methods based on trigonometric polynomials were also established for free form
curves and surfaces modeling (see, Han, Xi-An, Ma-Chen, Y. and Huang, X. [8], Dube, M. and Yadav, B., [5],
Han, X., ([6], [7]), Abbas M, Yahaya SH, Majid AA, Ali JM [1], I. J. Schoenberg [9], Yang L., Li J., Chen G,
([13], [14])). Liu, H., Li, L., and Zhang, D. [10] presented a study on class of TC-Bèzier curve with shape
trigonometric Bèzier curve with two shape parameters. Base on this idea, we have constructed the Rational
Quartic Trigonometric Bèzier curve with two shape parameters.
In this paper, rational quartic trigonometric Bèzier curve with two shape parameters is presented. The
purposed curve enjoys all the geometric properties of the traditional rational quartic Bèzier curve. The local control on the
shape of the curve can be attained by altering the values of the shape parameters as well as the weight. The curve exactly represents some quartic trigonometric curves such as the arc of an ellipse and the arc of a circle
and best approximates the ordinary rational quartic Bèzier curve. The paper is organized as follows. In section 2, the asis functions of the quartic trigonometric Bèzier curve with two shape parameters are established and the
properties of the basis function has been described. In section 3, rational quartic trigonometric Bèzier curves and
their properties are discussed. In section 4, By using shape parameter, shape control of the curves is studied and
explained by using figures. In section 5, the representation of ellipse and circle are given. In section 6, the
approximation of the rational quartic trigonometric Bèzier curve to the ordinary rational quadratic Bèzier curve
is presented.

II. Quartic Trigonometric Bèzier Basis Functions
In this section, definition and some properties of quartic trigonometric Bèzier basis functions with two shape
parameters are given as follows:

Definition 2.1: For two arbitrarily real value of $\lambda$ and $\mu$ where $\lambda, \mu \in [0,1]$ the following three functions of
t($t \in [0,1]$) are defined as quartic trigonometric Bèzier basis functions with two shape parameter $\lambda, \mu$:

\[
\begin{align*}
    b_0(t) &= \left(1 - \sin \frac{\pi t}{2}\right)\left(1 - \lambda \sin \frac{\pi}{2} t\right)^3 \\
    b_1(t) &= 1 - b_0 - b_2 \\
    b_2(t) &= \left(1 - \cos \frac{\pi t}{2}\right)\left(1 - \mu \cos \frac{\pi}{2} t\right)^3
\end{align*}
\] (2.1)
For $\lambda = \mu = 0$, the basis functions are linear trigonometric polynomials. For $\lambda, \mu \neq 0$, the basis functions are quartic trigonometric polynomials.

**Theorem 3.1:** The basis functions (2.1) have the following properties:

(a) **Nonnegativity:** $b_i(t) \geq 0$, $i = 0, 1, 2$.

(b) **Partition of unity:** $\sum_{i=0}^{2} b_i(t) = 1$

(c) **Monotonicity:** For a given parameter $t$, $b_0(t)$ is monotonically decreasing $\lambda$ and $\mu$ respectively; $b_2(t)$ is monotonically increasing for the shape parameters $\lambda$ and $\mu$ respectively.

\[ \begin{align*}
(1 - \sin \frac{\pi}{2} t) &\geq 0, (1 - \lambda \sin \frac{\pi}{2} t) \geq 0, \\
(1 - \lambda \sin \frac{\pi}{2} t)^3 &\geq 0, \sin \frac{\pi}{2} t \geq 0, \\
(1 - \cos \frac{\pi}{2} t) &\geq 0, (1 - \mu \sin \frac{\pi}{2} t) \geq 0, \\
(1 - \mu \cos \frac{\pi}{2} t)^3 &\geq 0, \cos \frac{\pi}{2} t \geq 0, \lambda \geq 0, \mu \geq 0.
\end{align*} \]

It is obvious that $b_i(t) \geq 0$, $i = 0, 1, 2$.

(b) $\sum_{i=0}^{2} b_i(t) = (1 - \sin \frac{\pi}{2} t) (1 - \lambda \sin \frac{\pi}{2} t)^3 + (1 - (1 - \sin \frac{\pi}{2} t) (1 - \mu \sin \frac{\pi}{2} t)^3 - (1 - \cos \frac{\pi}{2} t) (1 - \mu \cos \frac{\pi}{2} t)^3 = 1$.

**Fig.1.** shows the curves of the rational quartic trigonometric basis functions for $\lambda = \mu = 1$ (red solid), $\lambda = \mu = 0.5$ (green solid), $\lambda = \mu = 0$ (blue dashed).

**III. Rational Quartic trigonometric Bézier curve**

We construct the rational quartic trigonometric Bézier curve with two shape parameters as follows:

**Definition 3.1:** The Rational Quartic Trigonometric Bézier curve with two shape parameters is defined as:

\[ C(t) = \frac{b_0 P_0 + \nu b_1 P_1 + b_2 P_2}{b_0 + \nu b_1 + b_2} \quad (3.1) \]

$t \in [0,1], \lambda, \mu \in [0,1]$, where $P_i (i = 0, 1, 2)$ are the basis functions defined in (2.1) and $\nu$ is scalar, called that weight of function. We assume that $\nu \geq 0$. If $\nu = 1$, we get a non-rational trigonometric Bézier curve, since the denominator is identically equal to one.

The curve defined by (3.1) possesses some properties which can be obtained easily from the properties of the basis function.

**Theorem 3.1:** The Rational Quartic trigonometric Bézier curve (3.1) have the following properties:

(a) **End point properties:**

$C(0) = P_0$, $C(1) = P_2$,

$C'(0) = \frac{\pi}{2} (1 + 3\lambda)(P_1 - P_0)\nu$,

$C'(1) = \frac{\pi}{2} (1 + 3\mu)(P_2 - P_1)\nu$,

$C''(0) = \frac{\pi^2}{4} [\nu(2\lambda(2 + 3\lambda) + (1 - \mu)^2(1 + \mu) + 2(1 + 3\lambda)^2(\nu - 1))(P_1 - P_0) + (1 - \mu)^2(1 + \mu)(P_2 - P_0)]$,

$C''(1) = \frac{\pi^2}{4} [\nu(2\mu(2 + 3\mu) + (1 - \lambda)^2(1 + \lambda) + 2(1 + 3\mu)^2(\nu - 1))(P_2 - P_1) + (1 - \lambda)^2(1 + \lambda)(P_0 - P_2)]$,

(b) **Symmetry:** $P_0, P_1, P_2 and P_2, P_1, P_0$ define the same curve in different parametrizations, that is $C(t; \lambda, \mu; P_0, P_1, P_2) = C(1 - t; \mu, \lambda; P_0, P_1, P_2)$, $t \in [0,1], \lambda, \mu \in [0,1]$. 

\[ \]
Geometric invariance: The shape of the curve (3.1) is independent of the choice of coordinates, i.e., it satisfies the following two equations:

\[
C(t; \lambda, \mu; P_0 + q, P_1 + q, P_2 + q) = C(t; \mu, \lambda; P_0, P_1, P_2) + q
\]

\[
C(t; \lambda, \mu; P_0 \ast T, P_1 \ast T, P_2 \ast T) = C(t; \mu, \lambda; P_0, P_1, P_2) \ast T
\]

where \(q\) is an arbitrary vector in \(\mathbb{R}^2\) and \(\mathbb{R}^3\) and \(T\) is an arbitrary \(d \times d\) matrix, \(d = 2\) or \(3\).

Convex hull property: From the non-negativity and partition of unity of basis functions, it follows that the whole curve is located in the convex hull generated by its control points.

IV. Shape control of the rational quartic Trigonometric Bèzier curve

The parameters \(\lambda\) and \(\mu\) controls the shape of the curve (3.1). In figure 2, The rational quartic trigonometric Bèzier curve \(C(t)\) gets closer to the control polygon as the values of the parameters \(\lambda\) and \(\mu\) increases. In figure 2, the curves are generated by setting the values of \(\lambda = \mu = 0, v = 2\) (red solid lines) \(\lambda = \mu = 0.5, v = 3\) (blue dashed lines), \(\lambda = \mu = 1, v = 4\) (green solid lines).

V. The representation of Ellipse

**Theorem 5.1:** Let \(P_0, P_1\) and \(P_2\) be three control points on an ellipse with semi axes \(\sqrt{2a}\) and \(\frac{1}{\sqrt{2}}b\), by the proper selection of coordinates, their coordinates can be written in the form

\[
P_0 = \left( -\frac{a}{b} \right), P_1 = \left( 0 \right), P_2 = \left( \frac{a}{b} \right)
\]

Then the corresponding rational quartic trigonometric Bèzier curve with the shape parameters \(\lambda = \mu = 0\) with \(v = \frac{1}{2}\) and local domain \(t \in [t_1, t_2]\) represents arc of an ellipse with

\[
\begin{align*}
  x(t) &= a \left( \cos \frac{\pi}{2} t - \sin \frac{\pi}{2} t \right) \\
  y(t) &= \frac{1}{2} b \left( \sin \frac{\pi}{2} t + \cos \frac{\pi}{2} t - 1 \right)
\end{align*}
\]

Where \(0 \leq t_1, t_2 \leq 4\).

**Proof:** If we take \(\lambda = \mu = 0\), \(v = \frac{1}{2}\) and

\[
P_0 = \left( -\frac{a}{b} \right), P_1 = \left( 0 \right), P_2 = \left( \frac{a}{b} \right)
\]

into (3.1), then the coordinates of rational quartic trigonometric Bèzier curve are

\[
\begin{align*}
  x(t) &= a \left( \sin \frac{\pi}{2} t - \cos \frac{\pi}{2} t \right) \\
  y(t) &= \frac{1}{2} b \left( \sin \frac{\pi}{2} t + \cos \frac{\pi}{2} t - 1 \right)
\end{align*}
\]
According to theorem (5.1), if, $\overline{B} \mathcal{P} (1 + 1 = 2) = \overline{B} + \overline{P} + \overline{1} = \overline{1} + \overline{1} = \overline{1} + 1$.

Suppose $\lambda \overline{P} + \overline{0} = \overline{0} = \overline{1} + \overline{1} - \overline{1} = \overline{1} + \overline{1}$.

For simple computation, then $\overline{B} = \overline{P}_0 = \overline{B}_0, \overline{C} = \overline{P}_2 = \overline{B}(1)$ and

$B\left(\frac{1}{2}\right) - P_1 = \frac{1}{2(1 + v)} (P_0 - 2P_1 + P_2)$

For $\lambda = \mu$, $\overline{C} \left(\frac{1}{2}\right) - P_1 = \frac{(1 - \frac{1}{\sqrt{2}}) (1 - \frac{\lambda}{\sqrt{2}})}{(1 - \frac{1}{\sqrt{2}}) (1 - \frac{\lambda}{\sqrt{2}})^3} (P_0 - 2P_1 + P_2)$

Let $\left[ (1 - \sin \frac{\pi}{2} t) (1 - \lambda \sin \frac{\pi}{2} t)^3 \right]^{-1}$, then

$C\left(\frac{1}{2}\right) - P_1 = \frac{(P_0 - 2P_1 + P_2)}{2(1 + v) + kv}$

This gives the intrinsic equation

$$\frac{(x(t))^2}{2a} + \left(\frac{y(t) + \frac{1}{2} b}{\frac{1}{\sqrt{2}} b}\right)^2 = 1.$$  

It is an equation of an ellipse. Fig.3 shows the Ellipse.

**Corollary 5.2:** According to theorem (5.1), if, $a = \frac{1}{\sqrt{2}}, b = \sqrt{2}$ then the corresponding Rational Quartic trigonometric Bèzier curve with the shape parameter $\lambda = \mu = 0$ and local domain $t \in [0,4]$ represents arc of the circle. Fig.4 shows the Circle.

**VI. Approximability**

Control polygons play an important role in geometric modeling. It is an advantage if the curve being modeled tends to preserve the shape of its control polygon. Now we show the relation of the rational quartic trigonometric Bèzier curves and rational quadractic Bèzier curves with same control points. 

**Theorem 6.1:** Suppose $P_0, P_1$ and $P_2$ are not collinear; the relationship between rational quartic trigonometric Bèzier curves $C(t)$ (3.12) and rational quadratic Bèzier curve.

$$B(t) = \frac{\sum_{i=0}^{2} P_i B_i(t) w_i}{\sum_{i=0}^{2} B_i(t) w_i} \quad (6.1)$$

where $B_i(t) = \sum_{i=0}^{2} \binom{2}{i} (1 - t)^2 i^t$, $(i = 0,1,2)$ are the Bernstein polynomials and the scalar $w_i$ are the weights, with the same control points $P_i (i = 0,1,2)$ are as follows:

$C(0) = B(0), \quad C(1) = B(1)$

$C\left(\frac{1}{2}\right) - P_1 = \frac{2(1 + v)}{2(1 - v) + kv} \left( B\left(\frac{1}{2}\right) - P_1 \right)$

where $k = \left[ (1 - \sin \frac{\pi}{2} t) (1 - \lambda \sin \frac{\pi}{2} t)^3 \right]^{-1}$ with the assumption that $\lambda = \mu$.

**Proof:** We assume that $w_0 = w_2 = 1$ and $w_1 = v$ then the ordinary rational quadratic Bèzier curve (6.1) takes the form

$$B(t) = \frac{(1-t)^2 P_0 + 2(1-t) t P_1 + t^2 P_2}{(1-t)^2 + 2(1-t)tv + t^2}$$

By simple computation, then

$C(0) = P_0 = B_0, \quad C(1) = P_2 = B(1)$ and

$$B\left(\frac{1}{2}\right) - P_1 = \frac{1}{2(1 + v)} (P_0 - 2P_1 + P_2) \quad (6.3)$$

For $\lambda = \mu$, $C\left(\frac{1}{2}\right) - P_1 = \frac{(1 - \frac{1}{\sqrt{2}}) (1 - \frac{\lambda}{\sqrt{2}})}{(1 - \frac{1}{\sqrt{2}}) (1 - \frac{\lambda}{\sqrt{2}})^3} (P_0 - 2P_1 + P_2)$

Let $\left[ (1 - \sin \frac{\pi}{2} t) (1 - \lambda \sin \frac{\pi}{2} t)^3 \right]^{-1}$, then

$C\left(\frac{1}{2}\right) - P_1 = \frac{(P_0 - 2P_1 + P_2)}{2(1 + v) + kv}$
\[ C\left(\frac{1}{2}\right) - P_1 = \frac{2(1 + \nu)}{2(1 - \nu) + k\nu} B\left(\frac{1}{2}\right) - P_1. \]

Then (6.2) holds.

**Corollary 6.1:** The rational quartic trigonometric Bézier curve is closer to the control polygon than the rational quadratic Bézier curve if and only if \( \frac{\sqrt{\nu}(\sqrt{\nu}-1)^{1/3}}{(\sqrt{\nu}-1)^{1/3}} \leq \lambda, \mu \leq 1. \)

**Figure 5:** the relationship between the rational quartic trigonometric Bézier curve and rational quadratic Bézier curve.

**Corollary 6.2:** When \( \lambda = \mu = \frac{\sqrt{\nu}(\sqrt{\nu}-1)^{1/3}}{(\sqrt{\nu}-1)^{1/3}} \), the rational quartic trigonometric Bézier curve is closer to the rational quadratic Bézier curve, i.e. \( C\left(\frac{1}{2}\right) = B\left(\frac{1}{2}\right). \)

Fig.5 shows the relationship between the rational quartic trigonometric Bézier curve and rational quadratic Bézier curve. The quartic trigonometric Bézier curve is closer to rational quadratic Bézier curve. The rational quartic trigonometric Bézier curve (blue dashed) with parameter \( \lambda = \mu = \frac{\sqrt{\nu}(\sqrt{\nu}-1)^{1/3}}{(\sqrt{\nu}-1)^{1/3}} \) is analogous to ordinary quadratic Bézier curve (green solid).

**VII. Conclusion**

In this paper, we have presented the rational quartic trigonometric Bézier curve with two shape parameters. Each section of the curve only refers to the three control points. We can design different shape curves by changing parameters. The curve represent ellipse and circle when adjusting the control points and parameter value.

**References**


