



## NETWORK SIMPLEX METHOD USED TO TRANSPORTATION PROBLEM

Md. Mijanoor Rahman

Assistant Professor, Department of Civil Engineering, Presidency University, Gulshan-2, Dhaka, Bangladesh

Md. Kamruzzaman

Lecturer, Department of Civil Engineering, Presidency University, Gulshan-2, Dhaka, Bangladesh

**Abstract:** A Transportation problem (TP) with huge number of variables can be solved by Modified Distribution Method (MODIM) and Stepping Stone Method (SSM) both are simplex method which is used in operation research such as Traveling salesmen problem, Assignment problem, Network problem and many industrial management problem. Again a network simplex method is used Maxima -and Minimal Cost Flow problem (MMCFP) by using minimum spinning tree, optimality condition and dual solution. Here a Transportation problem (TP) has been taken and converted to Minimal Cost Flow problem (MCFP) then solved by network simplex method. An accurate result has been found by applying this method.

**Keyword:** Network Simplex Method (NSM), Minimal Cost Flow problem (MCFP), Transportation problem (TP), Network Flow, Optimality Condition.

### I. Introduction:

A rich history has for minimum cost flow problem in practical field. In the last few years, the Transportation, Traveling Salesman problem (TSP), Vehicle- Routing problem (VRP) and Maxima -and Minimal Cost Flow problem (MMCFP) are focusing for solving with well-known network problems Kantorovich [10], Hitchcock [8], and Koopmans [1947] was incompletely through and solved of the minimum cost flow problem. The first complete solution of specializing simplex algorithm for linear programming developed for the transportation problem by Dantzig [5]. Ravindra K. Ahuja, Thomas L. Magnant, James B. Orlin[1] presented decomposition-based simplex method for minimum cost network flows problem. When the conventional decomposition with subgradient optimization and Dantzig-Wolfe decomposition techniques were very slow then M. Babul Hasan and John F. Raffensperger [9] develop a decomposition-based pricing method for the solving of the large fishery model. A thesis work, An Algorithm and Its Computer Technique for Solving Game Problems Using LP method was also presented by H. K. Das and M. B. Hasan [6]. In the paper firstly we have converted from TP to MCFP and then applied NSM by adding artificial arc and solving dual simplex problem.

### II. Optimality Conditions: [1]

A. A feasible flow  $\xi$  is an optimum flow if and only if it admits no negative cost augmenting cycle.

Proof: Let  $\xi$  be any feasible solution and  $\xi^*$  be the optimum solution such that  $\xi^* = \xi$ . The augmenting cycle property implies that  $\xi^* - \xi$  can be decomposed into  $r$  augmenting cycles and the sum of the costs of these cycles equals  $c\xi^* - c\xi$ . If  $c\xi^* < c\xi$  then  $c\xi^* - c\xi < 0$  and therefore one of the cycles must have a negative cost. If all cycles in the decomposition of  $\xi^* - \xi$  are of non-negative cost then  $c\xi^* - c\xi \geq 0$  which implies  $c\xi^* \geq c\xi$ . But since  $\xi^*$  is optimum, we must have  $c\xi^* = c\xi$ , which means that we have obtained another optimum solution because  $\xi^* = \xi$ .

#### B. MCF Problem

The primal problem is

$$\text{Minimize } \sum_{(i,j) \in A} c_{ij} x_{ij}$$

$$\text{Subject to } \sum_{j:(i,j) \in A} x_{ij} - \sum_{j:(i,j) \in A} x_{ji} = \mathcal{G}_i \quad \text{for all } i \in N, \quad 0 \leq x_{ij} \leq u_{ij} \quad \text{for all } (i, j) \in A$$

**C. Dual Problem**

The dual variable  $\hat{h}_i$  with the flow for node  $i$  and  $\delta_{ij}$  is the upper bound constraint of arc  $(i, j)$ . The Dual Problem is then:

$$\text{Maximize } \sum \mathcal{G}_i \hat{h}_i$$

$$\text{Subject to } \hat{h}_i - \hat{h}_j - \delta_{ij} \leq c_{ij} \quad \text{for all } (i, j) \in A, \quad \delta_{ij} \geq 0, \quad \text{for all } (i, j) \in A$$

**D. Complementary Slackness Theorem: [4]**

The primal or dual constraint and corresponding primal or dual variable are satisfied the condition: If a primal constraint is satisfied then the corresponding dual variable equals zero. And if a dual variable is positive then the corresponding primal constraint is exactly satisfied.

Proof: The complementary slackness conditions for the MCFP  $x_{ij} > 0 \Rightarrow \hat{h}_i - \hat{h}_j - \delta_{ij} = c_{ij}$ ,  $\delta_{ij} < 0 \Rightarrow x_{ij} = u_{ij}$ ,

which equivalent to optimality conditions:  $x_{ij} = 0 \Rightarrow \hat{h}_i - \hat{h}_j \leq c_{ij}$ ,  $0 \leq x_{ij} \leq u_{ij} \Rightarrow \hat{h}_i - \hat{h}_j = c_{ij}$  and  $x_{ij} = u_{ij} \Rightarrow \hat{h}_i - \hat{h}_j \geq c_{ij}$

The reduced cost of arc  $(i, j)$ ,  $\bar{c}_{ij}$  is defined as:  $\bar{c}_{ij} = c_{ij} - \hat{h}_i + \hat{h}_j$

The optimality condition is giving following

1.  $x$  is feasible.
2. If  $\bar{c}_{ij} > 0$  then  $x_{ij} = 0$
3. If  $\bar{c}_{ij} = 0$  then  $0 \leq x_{ij} \leq u_{ij}$
4. If  $\bar{c}_{ij} < 0$  then  $x_{ij} = u_{ij}$

These optimality conditions are same to previously optimality conditions will be shown. Let  $x$  is flows corresponding  $\hat{h}$  node potentials are satisfying the dual optimality conditions. Let  $P$  be any cycle in the network, along which we can augment flow. If the direction of flow in any arc is opposite to the direction of flow augmentation in the cycle  $P$ , then for the purposes of this exposition, interchange that arc  $(i, j)$  with the opposite arc  $(j, i)$ . In this case  $x_{ij} > 0$  and thus  $c_{ij} \leq 0$ . Also, interchange  $c_{ij}$  with  $c_{ij} = -c_{ji}$ . If the flow direction in an arc is consistent with the direction of cycle flow augmentation, then  $c_{ij} \geq 0$  and let  $c_{ij} = c_{ji}$ .

Now for all arcs,  $(i, j) \in P, c_{ij}^* \geq 0$  so,

$$\sum_{(i,j) \in P} c_{ij}^* \geq 0 \quad \sum_{(i,j) \in P} c_{ij}^* = \sum_{(i,j) \in P} c_{ij} = \sum_{(i,j) \in P} (-\hat{h}_i + \hat{h}_j) = \sum_{(i,j) \in P} c_{ij}$$

Hence the network satisfying the dual optimality conditions is free of negative cost augmenting cycles. To show the converse, we again transform the solution. Replace all arcs  $(i, j)$  with  $x_{ij} = u_{ij}$ , with arcs  $(j, i)$  where the optimality conditions become

$c_{ij} = 0$  if  $0 \leq x_{ij} \leq u_{ij}$  and  $c_{ij} > 0$ , otherwise.

A basis structure is defined by partitioning the set of arcs,  $A$ , into three disjoint sets, as follows:

B: the set of basic arcs, which form a spanning tree of the network, L: the set of non basic arcs at their lower bounds, U: the set of non-basic arcs at their upper bounds.

We suppose the network contains no negative cost augmenting cycles. Given a basis structure (B, L, U) we compute node potentials as follows, fulfilling the condition that  $c_{ij} = 0$  for all basic arcs.

1. Set  $\hat{h}(\text{root}) = 0$
2. Find an arc  $(i, j) \in B$  for which one node potential is known and the other not. If  $\hat{h}_i$  is known, then set  $\hat{h}_j = \hat{h}_i - c_{ij}$ . If  $\hat{h}_j$  is known, then set  $\hat{h}_i = \hat{h}_j + c_{ij}$
3. Repeat (2) until all  $\hat{h}$  are known.

All  $\hat{h}$ -values will be found since there are  $(n-1)$  arcs in B, and they form a spanning tree. Suppose there is some arc  $(l, k) \in L$  or  $U$  for which  $c_{ij} < 0$ . We can add basic arcs to form a cycle  $W$  with arc  $(l, k)$  such

that.

$$\sum_{(i,j) \in W} c_{ij} = \sum_{(i,j) \in W} \bar{c}_{ij}$$

$$+ \sum_{(i,j) \in W} (-h_i + h_j) = \bar{c}_{ik} < 0$$

Hence  $\sum_{(i,j) \in W} \bar{c}_{ij} < 0$  and thus  $W$  is a negative cost augmenting cycle. (It is an augmenting cycle by virtue of the definition of all arcs  $(i, j) \in B$  and the transformation of arc  $(1, k)$ ). Hence by contradiction, if the network contains no negative cost augmenting cycles, then it fulfills the dual (reduced cost) optimality conditions, so the two sets of optimality conditions are equivalent.

### III. The network simplex method for solving transportation problem:

Solution of the following transportation problem by network simplex method [4]

	1	2	3	$s_i$
1	4	7	5	30
2	2	4	3	20
$d_i$	15	10	25	

Table-1

Solution: The above problem written in the form of network flows.

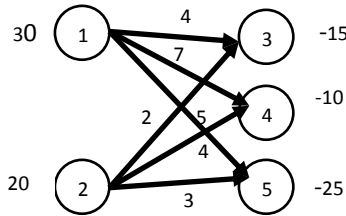


Figure: 1

Adding artificial arc to node 3 and cost is 0.

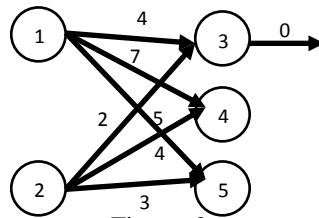


Figure: 2

Computing the values of the Basic variables:

Adding the artificial arc to node 3, suppose that we select the feasible basis given by the sub graph in Figure 1.

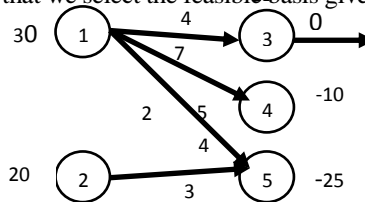


Figure: 3

The basic system of equation to be solved is  $Bx = b$ . Where  $x_5$  is the artificial variable associated with node 3

node	(2,5)	(1,5)	(1,4)	(1,3)	root-arc	b
2	1	0	0	0	0	20
5	-1	-1	0	0	0	-25
4	0	0	-1	0	0	-10
1	0	1	1	1	0	30
3	0	0	0	-1	1	-15

Table-2

Taking advantage of the lower triangular structure of the basis matrix, we may iteratively solve for the basic variables. From the top equation  $Bx = b$ ,  $x_{25} = 20, x_{15} + x_{25} = 25, x_{15} = 5, x_{14} = 10$  and  $x_{13} + x_{14} + x_{15} = 30$  or  $x_{13} = 15$ . Finally  $x_5 = -15 + x_{13} = 0 \Rightarrow x_{25} = 20, x_{13} = 15, x_{14} = 10, x_{15} = 5, x_5 = 0$  and thus the basis is feasible. These same computations can be made directly on the graph in Figure-4 as follows.

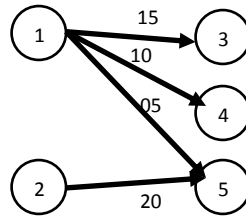


Figure: 4

Computing Dual Variables  $w$  and  $z_{ij} - c_{ij}$ :

Let the dual simplex multiplier  $W = [w_2 \ w_5 \ w_4 \ w_1 \ w_3]$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, W = [w_2 \ w_5 \ w_4 \ w_1 \ w_3], C = [3 \ 5 \ 7 \ 4 \ 0]$$

From  $WA = C$  we will get, if  $w_3 = 0$  then  $w_1 = 4, w_4 = -3, w_5 = -1, w_2 = 2$

To compute  $z_{ij} - c_{ij}$  for non basic  $(i, j)$ , we apply the definition.

$$z_{ij} - c_{ij} = wa_{ij} - c_{ij} = w(e_i - e_j) - c_{ij} = w_i - w_j - c_{ij} \quad z_{23} - c_{23} = w_2 - w_3 - c_{23} = 2 - 0 - 2 = 0$$

$$z_{24} - c_{24} = w_2 - w_4 - c_{24} = 2 + 3 - 4 = 1$$

Determining the Exiting Column and Pivoting:

If the cycle method is applied to compute  $z_{ij} - c_{ij}$  for a non basic arc then we have to identify the pivot process. Here  $x_{24}$  is a candidate to enter the basis

Here the basic tree together with arc (2, 4) is following.

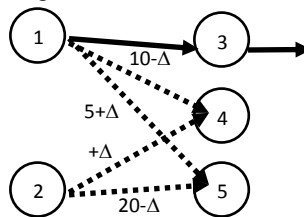


Figure: 5

Since the non basic variable  $x_{24}$  is entering to basic and the above figure has shown that the value of  $x_{14}$  &  $x_{25}$  are decreased by  $\Delta$  and the minimum value of  $x_{14}$  &  $x_{25}$  is 10. So the value of  $\Delta$  is 10 and  $x_{14}$  will leave the basis. The new feasible solution is

$$x_{25} = 10, x_{13} = 15, x_{24} = 10, x_{15} = 15, x_3 = 0$$

Computing Dual Variables  $W$  and  $z_{ij} - c_{ij}$ :

Let the dual simplex multiplier  $W = [w_2 \ w_5 \ w_4 \ w_1 \ w_3]$

$$w_2 - w_4 = c_{24} = 4 \Rightarrow w_4 = 2 - 4 = -2, \quad w_1 - w_5 = c_{15} = 5 \Rightarrow w_5 = -5 + 4 = -1 \quad w_2 - w_5 = c_{25} = 3 \Rightarrow w_2 = 2,$$

$$w_1 - w_3 = c_{13} = 4 \Rightarrow w_1 = 4, \quad w_3 = 0$$

To compute  $z_{ij} - c_{ij}$  for non basic  $(i, j)$ , we apply the definition.

$$z_{ij} - c_{ij} = wa_{ij} - c_{ij} = w(e_i - e_j) - c_{ij} \quad z_{23} - c_{23} = w_2 - w_3 - c_{23} = 2 - 0 - 2 = 0$$

$$z_{12} - c_{12} = w_1 - w_2 - c_{12} = 4 - 2 - 7 = -5$$

Since all  $z_{ij} - c_{ij} \leq 0$ , so the solution is optimal and the optimal solution is

$$x_{25} = 10, x_{13} = 15, x_{24} = 10, x_{15} = 15$$

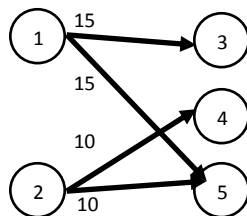


Figure: 6

#### IV. Conclusion

In this work we reviewed the Transportation Problem (TP) and Network Flow Problem. It has been shown that Solution of TP by general simplex method is possible but very time consuming where as especial methods like SSM & MODI solves the problem very easily but considering transportation problem as network flow problem and then solving it Network Simplex algorithm is more convenient.

#### References:

- [1] Ahuja, R. K., Magnanti, T. L., and Orlin, J. B., Network Flows Theory, Algorithms, and Applications, Handbooks in Operations Research and Management Science, Prentice hall, upper saddle river, New Jersey 07458, (1989), pp-402-460.
- [2] Ahuja R. K., Magnanti T. L., Orlin J. B., Reddy M.R., Applications of Network Optimization, (1992), pp.2 5-31.
- [3] Ahuja, R.K. and Orlin J.B., Improved Primal Simplex Algorithms for the Shortest Path, Assignment and Minimum Cost Flow Problems (1988).
- [4] Bazaraa, M. S., Jarvis, J. J., and Sherai, H. D., Linear Programming and Network Flows, 3rd Edition. John Wiley and Sons, New York, NY (2005).
- [5] Dantzig, G.B., Application of the Simplex Method to a Transportation Problem. In T.C. Koopmans (ed.). Activity Analysis of Production and Allocation, John Wiley & Sons, Inc (1951).
- [6] Das H.K. and Hasan M. B., A Generalized Computer Technique for Solving Unconstrained Non-Linear Programming, Dhaka Univ. J. Sci. 61(1), (2013): 75-80.
- [7] Das H.K. and Hasan M. B., An Algorithm and Its Computer Technique for Solving Game Problems Using LP method, IJBAS-IJENS Vol: 11 (2011) No: 03
- [8] Hitchcock, F. L. The Distribution of a Product from Several Sources to Numerous Facilities. Math. Phys. 20, (1941) pp.224-230.
- [9] Hasan M. B. and Raffensperger J. F., A Decomposition-Based Pricing Method for Solving a Large-Scale MILP Model for an Integrated Fishery, Hindawi Publishing Corporation, (JAMDS), Vol 2007, pp-10
- [10] Kantorovich, L. V. 1939, Mathematical Methods in the Organization and Planning of Production. Publication House of the Leningrad University, 68 pp. Translated in Mf in. Sci. 6(1960), pp366-422.