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An Open type Mixed Quadrature Rule using Fejer and Gaussian Quadrature Rules

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Abstract: *A mixed quadrature rule of higher precision for approximate evaluation of real definite integrals has been constructed. The analytical convergence of the rule has been studied. The relative efficiencies of the proposed mixed quadrature rules have been compared with the help of suitable test integrals. The error bound has been determined asymptotically.*

Keywords: *Fejer's second quadrature rule, mixed quadrature rule.*
2000 Mathematics Subject Classification: 65D30, 65D32

I. Introduction

Apart from two types of basic quadrature rules such as:

1. Newton-Cotes type of quadrature rules
2. Gaussian type of quadrature rules,

other effective rules such as: Clenshaw-Curtis and Fejer's type of quadrature rules are available in literature. Newton-Cote type of quadrature rules are based on approximation of the integrand by using simple interpolation such as: Lagrange's interpolation, where as Gaussian quadrature namely Gauss-Legendre quadrature is based on the approximation of the integrals by orthogonal polynomials such as: Legendre polynomials. Again both Clenshaw-Curtis and Fejer's type of quadrature rules are based on the approximation of the integrands by orthogonal polynomials such as: Chebyshev polynomials. Mostly Newton-Cotes rules are of closed type. Though open type Newton-Cotes rules are available, the Gaussian quadrature rules, itself is open, more powerful and efficient. This is because n point Newton-Cotes rules has precision n or $n-1$ according as n is odd or even. Where as n point Gaussian rule has precission $2n-1$.

On the other hand Fejer's quadrature rule is an open type rule. It has been tested that it is better than Newton-Cotes rule of same precision. This inspires us to form an open type mixed quadrature rule blending two open type given rules such as: Gauss-Legendre and Fejer's type of quadrature rules. The formulation of mixed quadrature rules was first coined by R. N. Das and G. Pradhan[**1**]. Many author's [**1-5, 14**] have produced different mixed quadrature rules. These mixed quadrature rules are mostly closed type rules.

In mixed quadrature rule, linear/ convex combination of two quadrature rules or more rules of equal precisions is taken to produce a new type of quadrature rule of higher precision. Though in literature we find methods such as Konord extension method[**7, 10, 11**] an d Richardson's extrapolation method[**12]** for precision enhancement, the mixed quadrature method is very simple and easy to compute as no additional evaluation of function is required while integrating the integral by this rule.

Due to above facts, In this paper, we get motivation for successfully forming an open type mixed rule of precision seven taking linear combination of Fejer's second and Gaussian 3-point quadratue rules, each of precision five. The mixed quadrature rule so found has been tested and compared with its constituent rules by computing numerically five tested integrals. The results have been tabulated in Table-4.1.

II. Construction of the Mixed Quadrature Rules of Precision Seven

Expressing the integrand in terms of Chebyshev polynomials one can derive Fejer's second $n-$point quadrature rule[9] as

$$\int_{-1}^{1} f(x)dx \approx R_{2Fn}(f) = \sum_{k=1}^{n} w_k\, f\left(Cos\frac{k}{n+1}\pi\right)$$

Where $w_k = \frac{4\,Sin\theta_k}{n+1}\sum_{m=1}^{\left[\frac{(n+1)}{2}\right]}\frac{Sin(2m-1)\theta_k}{2m-1}$

Taking $n=5$ one can write 5-point Fejer's second rule as

$$I(f) = \int_{-1}^{1} f(x)dx \approx R_{2F5}(f) = \frac{2}{45}\left[7f\left(\frac{\sqrt{3}}{2}\right) + 9f\left(\frac{1}{2}\right) + 13f(0) + 9f\left(-\frac{1}{2}\right) + 7f\left(-\frac{\sqrt{3}}{2}\right)\right]. \qquad (2.1)$$

We have well known the Gaussian Legendre 3-point rule:

$$I(f)=\int_{-1}^{1}f(x)dx \approx R_{GL3}(f)=\frac{1}{9}\left[5f\left(\sqrt{\frac{3}{5}}\right)+8f(0)+5f\left(-\sqrt{\frac{3}{5}}\right)\right]. \quad (2.2)$$

Each of the rules (2.1) and (2.2) is of precision five.

Let $E_{2F5(f)}$ and $E_{GL3}(f)$ denote the errors in approximating the integrals $I(f)$ by the rules (2.1) and (2.2) respectively.

Then

$$I(f)=R_{2F5}(f)+E_{2F5}(f) \quad (2.3)$$

$$I(f)=R_{GL3}(f)+E_{GL3}(f) \quad (2.4)$$

Using Maclurin's expansion of functions in equations (2.1) and (2.2) we have

$$E_{2F5}(f)=I(f)-R_{2F5}(f)$$

$$=\frac{1}{67200}f^{vi}(0)+\frac{1}{1814400}f^{viii}(0)+\ \dots \quad (2.5)$$

and $E_{GL3}(f)=I(f)-R_{GL3}$

$$=\frac{8}{126000}f^{vi}(0)+\frac{88}{45360000}f^{viii}(0)+\ \dots \quad (2.6)$$

Now multiplying the equations (2.3) and (2.4) $-\frac{1}{8}$ and $\frac{8}{15}$ respectively, and then adding the resulting equations, we have

$$I(f)=\frac{1}{49}[64R_{2F5}(f)-15R_{GL3}(f)]+\frac{1}{49}[64E_{2F5}(f)-15E_{GL3}(f)]$$

or $I(f)=R_{2F5GL3}(f)+E_{2F5GL3}(f)$ (2.7)

Where $R_{2F5GL3}(f)=\frac{1}{49}[64R_{2F5}(f)-15R_{GL3}(f)]$

and $E_{2F5GL3}(f)=\frac{1}{49}[64E_{2F5}(f)-15E_{GL3}(f)]$

Hence

$$R_{2F5GL3}(f)=\frac{1}{2205}\left[896f\left(-\frac{\sqrt{3}}{2}\right)-375f\left(-\sqrt{\frac{3}{5}}\right)+1152f\left(-\frac{1}{2}\right)+1064f(0)+1152f\left(\frac{1}{2}\right)-375f\left(\sqrt{\frac{3}{5}}\right)+896f\left(\frac{\sqrt{3}}{2}\right)\right]. \quad (2.8)$$

This is the desired mixed quadrature rule of precision seven for the approximate evaluation of $I(f)$.

The truncation error associated to this rule is given by

$$E_{2F5GL3}(f)=\frac{1}{49}[64E_{2F5}(f)-15E_{GL3}(f)]$$

or $E_{2F5GL3}(f)=\frac{1}{5!\times 68600}f^{viii}(0)+\ \dots$ (2.9)

III. Error Analysis

An asymptotic error estimate and an error bound of the rule(2.8) are given in theorems$(3.1(a))$ and $(3.1(b))$ respectively.

Theorem $3.1(a)$: Let $f(x)$ be a sufficiently differentiable function in the closed interval $[-1,1]$. Then the error $E_{2F5GL3}(f)$ associated with the rule $R_{2F5GL3}(f)$ is given by

$$|E_{2F5GL3}(f)|\approx\frac{1}{5!\times 68600}\left|f^{viii}\right|(0).$$

Proof: From eq.(2.7)

$$E_{2F5GL3}(f)=R_{2F5GL3}(f)+E_{2F5GL3}(f)$$

Where $R_{2F5GL3}(f)=\frac{1}{49}[64R_{2F5}(f)-15R_{GL3}(f)]$

and $E_{2F5GL3}(f)=\frac{1}{49}[64E_{2F5}(f)-15E_{GL3}(f)]$

Hence $E_{2F5GL3}(f)=\frac{1}{5!\times 68600}f^{viii}(0)+\ \dots$

So, $|E_{2F5GL3}(f)|\approx\frac{1}{5!\times 68600}\left|f^{viii}(0)\right|$.

Theorem $3.1(b)$: The bound for the truncation error $E_{2F5GL3}(f)=I(f)-R_{2F5GL3}(f)$ is given by

$$|E_{2F5GL3}(f)|\leq\frac{M}{51450}|(\eta_2-\eta_1)|;\ \eta_1,\eta_2\in[-1,1]$$

Where $M=\underset{-1\leq x\leq 1}{Max}\left|f^{viii}(x)\right|$.

Proof: We have $E_{2F5}(f)\approx\frac{1}{67200}f^{vi}(\eta_1),\eta_1\in[-1,1]$

$$E_{GL3}(f) \approx \frac{8}{175 \times 6!} f^{vi}(\eta_2), \eta_2 \in [-1, 1]$$

Hence $E_{2F5GL3}(f) = \frac{1}{49}[64E_{2F5}(f) - 15E_{GL3}(f)]$

$$= \frac{1}{51450}\int_{\eta_1}^{\eta_2} f^{vii}(x)dx, \quad \eta_1 < \eta_2$$

From this we have

$$|E_{2F5GL3}(f)| = \left|\frac{1}{51450}\int_{\eta_1}^{\eta_2} f^{vii}(x)dx\right| \leq \frac{1}{51450}\int_{\eta_1}^{\eta_2}\left|f^{vii}(x)dx\right|$$

$$\therefore |E_{2F5GL3}(f)| \leq \frac{M}{51450}|(\eta_2 - \eta_1)|$$

Where $M = \underset{-1\leq x\leq 1}{Max}\left|f^{viii}(x)\right|$.

This shows that the error bound as η_1, η_2 are unknown points in $[-1, 1]$. Also, it gives the error in approximation will be minimize if the points η_1, η_2 are very close to each other.

Corollary: The error bounds for the truncation error $E_{2F5GL3}(f)$ is given by

$$|E_{2F5GL3}(f)| \leq \frac{2M}{51450}.$$

Proof: From $3.1(b)$, we have

$$|E_{2F5GL3}(f)| \leq \frac{M}{51450}|(\eta_2 - \eta_1)|; \ \eta_1, \eta_2 \in [-1, 1]$$

Where $M = \underset{-1\leq x\leq 1}{Max}\left|f^{viii}(x)\right|$

Choosing $|(\eta_2 - \eta_1)| \leq 2$, we have

$$|E_{2F5GL3}(f)| \leq \frac{2M}{51450}.$$

IV. Numerical Verifications

Comparison of the mixed quadrature rule with 5-point Fejer's second rule and Gauss-Legendre 3-point rule in approximation of some real definite integrals.

Table-4.1

Integrals	Exact value	Approximate value			Error Approximated		
		$R_{2F5}(f)$	$R_{GL3}(f)$	$R_{2F5GL3}(f)$	$E_{2F5}(f)$	$E_{GL3}(f)$	$E_{2F5GL3}(f)$
$\int_{-1}^{1}(\ln(x^2+1))dx$	0.5278870147	0.5267202238	0.5222262547	0.528095924	1.1667909 $\times 10^{-3}$	5.66076 $\times 10^{-3}$	2.089093 $\times 10^{-4}$
$\int_{-1}^{1} e^x Cosxdx$	1.933421496	1.933412684	1.933390469	1.933419484	0.599294656	3.1027 $\times 10^{-5}$	2.012 $\times 10^{-6}$
$\int_{-1}^{1}\frac{dx}{1+Cosx}$	1.09260498	1.092562943	1.092434788	1.092602237	4.2037 $\times 10^{-5}$	1.70192 $\times 10^{-4}$	2.743 $\times 10^{-6}$
$\int_{-1}^{1} Sinh(x^2+1)\,dx$	3.701158418	3.696798227	3.684143231	3.700672204	4.360191 $\times 10^{-3}$	0.017015187	4.86214 $\times 10^{-4}$
$\int_{0}^{3} e^x \ln(x^2+2)dx$	35.88047234	35.87568054	35.86068652	35.88027053	4.7915 $\times 10^{-3}$	0.01978582	2.6704 $\times 10^{-4}$

V. Conclusions

From the above table, we observed that the new type of mixed quadrature rule $(R_{2F5GL3}(f))$ formed in this paper gives better approximation as compared to the constituent rules such as: 5-point Fejer's second rule$(R_{2F5}(f))$and Gauss-Legendre 3-point rule$(R_{GL3}(f))$. Hence, we conclude that the mixed quadrature rule is more preferred than constituent basic rules.

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