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Inertial Effects on Hydrodynamic Convection in a Passive Mushy Layer

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***Abstract**: This paper deals with the analytical investigation of convection in a passive mushy layer under the influence of inertial effects, formed during the solidification of a binary alloy melt. The mush-solid interface is at a eutectic temperature while the mush-liquid interface is impervious. Analytical solution for the basic state, zeroth and first-order systems are determined by a perturbation technique through normal mode approach. Higher - order solutions are obtained by applying the solvability conditions. The profiles of total Rayleigh number, vertical velocity, temperature and local solid - fraction are presented for the case of constant permeability. It is found that the inertial effect has a stabilising influence in the convection in a mushy layer, which in turn facilitates the suppressions of chimney formation which has catastrophic effects on the internal structure and the quality of the solid formed. The present analytical results are extremely sensitive to the far-field temperature and are in excellent agreement with the available numerical results in the limiting cases.*

***Keywords**: Inertial effects, Mushy layer, Chimney, Passive layer, Dendrites, far field temperature.*

I.Introduction

During the solidification process of aqueous solutions, metallic alloys and magmas the interface between the liquid and the solid becomes dendritic due to the morphological instability. Thus a mushy region consisting of a partially solidified melt and the fluid, will be formed and at the transition point, convection prevails. The same phenomenon will be observed even in the case of pure materials when they are cooled to a Meta stable state which is much below its melting temperature. The interesting phenomenon that occurs within the solidifying melt is the formation of chimneys which are narrow dendritic-free cylindrical regions or narrow cylindrical region of zero solid fractions and are very much similar to the imperfections called ‘freckles’ that appear in the casting of metallic alloys[1][2].

During the evolution of the solidification process three stages viz., finger, plume and chimney convections are observed[3][4]. The formation of chimneys during the solidification of a binary or a multicomponent alloy constitutes the following evolution procedure. Actually, during the solidification process the solidification front or the interface between the solid and the liquid becomes highly dendritic due to the morphological instability. As a consequence there will be a formation of region called ‘ mushy layer’ consisting of a partially solidified melt, the dendritic structure of which is quite complex[5]. Then the system becomes unstable due to the density gradient that results from the rejected materials and there will be a transition to convection. Further, as a result of the interaction of the thermal fields and the generated convective motions, chimneys which are responsible for the imperfections in the resulting solid will be formed [1][6][7][8][9].

Especially in metallurgy, dynamics of sea and geophysics, the mechanism and the process of formation of chimneys which spoil the quality, physical properties and the internal structure of the resulting solid, are important study areas[10][11][12]. In the past three decades the study pertaining to the development of different convective models and analysis for the case of convection in mushy layers has attracted researchers [13][8]. The works connected with the formulation of the governing equations in the study of convection in mush layers, the development of mathematical models and the solution procedure are available[14][15] .

Linear and weakly nonlinear convective instability in a mushy layer has been studied by quite a number of researchers under different types of assumptions and approximations[15][6][16][17]18]19][20]. Quite a number of works on convective flow in a mushy layer is available. A detailed review on convection in mushy layers is given by[6][21]. Recently [22][23][24] have applied weakly nonlinear evolution approach to study two-dimensional convective motions in a mushy layer with impermeable solidification front under different situations.Finally[25] have studied numerically the effects of inertia on convection in a mushy layer with constant permeability. The main objectives behind these studies are to study the history of the solidification process which is responsible for the formation of the pure solid and to suppress the formation of freckles which have catastrophic effects on the internal structure of the resulting solid.

A thorough survey of the literature pertaining to the subject reveals that no analytical work is available for the case of convection in a mushy layer with and without constraints. Therefore the present analytical study is made to study the effects of inertia on convection in a mushy layer under near eutectic temperature, large far-field temperature and latent heat. The boundaries are assumed to be impervious so that the Darcy's equation is valid. Marginal stability curves and the profiles of the total vertical velocity, temperature and local solid fraction are presented for the experimental values of the governing parameters.

II. Mathematical formulation

The physical configuration consists of a horizontal mushy layer formed during the solidification of a binary alloy as shown in fig 1. The process of uniform cooling from below of the system results in the upward advancement of the solid – mush interface with a constant solidification speed V_0 .In other words, the mushy layer is sandwiched between the solid and the liquid regions. The study is carried out in a moving frame of reference.

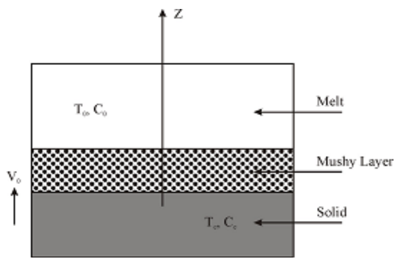


Figure 1. The schematic diagram of the physical system

Following are the assumptions made for the study:

i. The top and the bottom boundaries of the mushy layerare assumed to be isothermal, non-deformable and impermeable to the fluid flow, so that the mushy layer is kept dynamically isolated from the other components of the system [17].

ii. The solidification front is moving upwards with a velocity V_0 relative to the solid formed and the solid dendrites within the mushy layer. This makes the basic state to be steady.

iii. The temperature T and the composition C of the liquid in the mushy layer are required to satisfy a linear liquidus relationship $T = T_0(C_0) + \Gamma(C - C_0)$, where Γ is a constant. The liquid is assumed to be Newtonian with a linearized equation of state$\rho = \boldsymbol{\rho_0}[1 + \beta(C - C_0)]$ where ρ is the density of the liquid and$\rho_0$is a reference density, $\boldsymbol{\beta} = \boldsymbol{\beta}^* - \boldsymbol{\alpha}^*\Gamma$, $\boldsymbol{\alpha}^*$and $\boldsymbol{\beta}^*$are constant exponent coefficients for heat and solute respectively.

iv. First, following [17] we study in the limit in which the thickness of the mushy layer is much less than the diffusion length scale by letting $\delta \leq 1$.

v. However that a key implication of the near-eutectic approximations $C = O(\delta^{-1})$ is that the solid fraction is small and hence the permeability is uniform to the lowest order. As a consequence, we follow[17] and expand the permeability in terms of the small solid fraction :
$K(\Phi) = 1 + K_1(\Phi) + K_2(\Phi^2) + K_3(\Phi^3)$+-----------

Where on physical grounds, we demand that K_1, K_2 K_3,….etc arenon-negative.

Under the above assumptions and approximations the governing equations of the system are Conservation of momentum, Conservation of mass, Conservation of heat and solute :

$$\frac{\mu}{\pi}\vec{q} + \frac{1}{1+\phi}\left(\partial_t + \frac{\vec{q}\nabla}{1-\phi}\right)\vec{q} = -\nabla p - (\rho - \rho_0)\, g\, \vec{k} \tag{1}$$

$$\nabla.\vec{q} = 0 \tag{2}$$

$$\frac{\partial T}{\partial t} + (\vec{q}.\nabla)\, T = k\, \nabla^2 T + \frac{l_h}{\gamma}\frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{t}} \quad (3$$

$$\chi\, \frac{\partial C}{\partial t} + (\vec{q}.\nabla)\, C = (C - C_0)\frac{\partial \Phi}{\partial t} \tag{4}$$

where
Φ = 1- χ : Local solid fraction, χ – Local liquid fraction, P – Dynamic pressure, $\pi=\pi(\Phi)$, permeability is a function of the local solid fraction , μ :Dynamic viscosity, t, T, k, l_h, γ are time, temperature, thermal diffusivity, specific heat, latent heat/unit mass. ,Cs: Composition of the solid phase ,C_0: Composition of the liquid phase ,ρ, ρ_0 : densities , $\vec{g}$ = (0,0,g) acceleration due to gravity.,$\pi(0)$: The reference permeability., $\vec{q}$ = u i + vj +w k ,d: the mushy layer thickness, $\vec{q}$ is the Darcy velocity vector and (u,v,w) are the horizontal and vertical components of $\vec{q}$, i, j, k : unit vectors along the x, y and z axes.
The boundary conditions are:
$T = T_e$, w =0 on z =0,
$T = T_0$, w = 0, Φ =0 on z = d. (5)
Here T_0 is the temperature at the mush-liquid interface (z = d) and Ts and Cs are the eutectic temperature and concentration at the mush-solid interface (z =0).

The dimensionless equations using the scales mentioned below are:

$$\mathbf{K}\vec{q} + \frac{I^*}{1-\phi}[(\partial_t - \partial_z) + (\frac{\vec{q}.\nabla}{1-\phi})\vec{q} = -\nabla P - R.\Theta.\hat{k} \quad (6)$$

$$(\partial_t - \partial_z)(\theta - S\Phi) + (\vec{q}.\nabla)\,\theta = \nabla^2\theta, \quad (7)$$

$$(\partial_t - \partial_z)\,((1-\Phi)\,\theta + C\,\Phi) + (\vec{q}\cdot\nabla)\theta = 0, \quad (8)$$

$$\nabla.\vec{q} = 0 \quad (9)$$

In this paper the case of constant permeability i.e., K =1 is studied.
The scales used for the non- dimensionalisation of the governing equations are

$\vec{q} = V_0 .. \vec{q}^{\,*}$(velocity)

$(x,y,z) = \frac{k}{v_0}(x^*,y^*,z^*)$ (length)

$t = \frac{k}{v_0 2}t^*$ (time)

$p = \frac{k}{\mu\pi(0)}p^*$(pressure)

$K = \frac{\pi(0)}{\pi(\emptyset)}$

(10)

The dimensionless parameters appearing in the problem are:

$R = \frac{\beta_{\Delta C.\pi(0)g}}{\mu V_0}$ (Rayleigh number)

β is the volume expansion coefficient of the combined heat and solute[6].

$S= \frac{l_h}{\gamma\Delta T}$: Stefan Number

$C=\frac{C_0-C_e}{\Delta c}$: Concentration Ratio

$I^* = \frac{\pi(0)}{\mu}$: Inertia parameter

where
$\Delta T = T_0 - Te$, $\Delta C = C_0 - C_e$, $\pi(0)$ is the reference of π
T : Time variable
Φ : Local solid fraction
$\boldsymbol{v}$ **:** Kinematic viscosity
$\vec{V}_0$: Velocity of the solidification front (11)

$\theta = \frac{T-T_0}{\Delta T} = \frac{C-C_0}{\Delta C}$ (The non-dimensional temperature or composition)

The dimensionless boundary conditions are:
The non-dimensional boundary conditions at the upper boundary z = d correspond to an impermeable (rigid) flat boundary with zero solid fraction (i.e., φ =0).Thus we have
θ = -1, w = 0 @ z = 0
θ = 0, φ = 0, w = 0 @ z = δ (12)
where $\delta = \frac{dV0}{K}$ is the growth peclet number and also the dimensionless thickness of the mushy layer

III. Method of solution

The method of solution constitutes two stages viz., basic state analysis and linear stability analysis. In the case of variable permeability K is a function porosity or the local solid fraction and is expressed as K(x,y,z,t) = $\frac{\pi\,(0)}{\pi\,(\varphi)}$ which is similar to the Kozney - Carman relation[6]. The case of constant permeability corresponds to the decoupling of permeability and porosity. In that case

$K(\varphi) = [1- \varphi\,(x,y,z,t)]^{n}$, n = 0 (13)

As discussed earlier, the physical configuration is such that it consists of a mushy layer. T_e is the eutectic temperature at which the lower mush – solid interface is maintained in which T_∞ is the temperature of the liquid far above the mushy layer. Further $T_0(C_0)$ is the liquidus temperature of the alloy such that $T_\infty > T_0(C_0)$ and the mushy layer is assumed to be in a state of thermodynamic equilibrium so that

$T = T_0(C_0) + \Gamma(C - C_0)$ (14)

In this work the attention is focused on the prediction of inertial effects on convection in a mushy layer with constant permeability

IV. Basic state analysis

The basic state corresponds to the steady motionless state in which $\vec{q}$ = 0 and $\frac{\partial}{\partial t}$ = 0. Thus we have the following set of equations :

Conservation of solute:

$(1 - \varphi_b)\, D\, \theta_b + D\, \varphi_b\, (C - \theta_b) = 0$ (15)

Conservation of heat :

$D^2 \theta_b + D\, \theta_b - S\, D\, \varphi_b = 0$ (16)

Conservation of momentum

$\frac{dpb}{dz} - k.\theta_b = 0$ (17)

The boundary conditions are

$\theta_b = -1$ @ z = 0

$\theta_b = \theta_\infty$, $\varphi_b = 0$ @ z = δ (18)

where θ_∞ is the far-field temperature

Here we take $\varphi_b = \delta\varphi_{b0}$, $\theta_b = \theta_{b0}$ (19)

Substituting (19) in (15) and (16) we get

$(1 - \delta\varphi_{b0})\, D\, \theta_{b0} + \delta\, D\, \varphi_{b0}\, (C - \theta_{b0}) = 0$ (20)

$D^2 \theta_{b0} + D\, \theta_{b0} - \delta\, S.D\, \varphi_{b0} = 0$ (21)

Collecting the terms of O (δ^0) and O(δ) from (20) and (21) and solving the differential equations by using (18), we get the basic state solutions as

$\theta_{b0} = (- 1 + e\, \theta_\infty) - e\, \theta_\infty .\, e^{-z}$ (22a)

Or

$\theta_{b0}\ \ - 1 + e\, \theta_\infty - e\, \theta_\infty.(1+z)$

$\theta_b = \theta_{b0} = - 1 + e.\theta_\infty\, z$

Finally, we can write

$\theta_b = C_1 + C_2\, Z$ and $\emptyset_b = C_3 + C_4\, Z$ (22b)

where $C_1 = -1$, $C_2 = e\, \Theta_\infty$. .$C_3 = \frac{c+s+1}{s}$, $C_4 = \frac{e\, \Theta_\infty.}{s}$

V. Linear stability analysis

As discussed earlier the analysis consists of two stages viz., the basic state analysis and the linear stability analysis. For this purpose, we consider the expansion of the dependent variable Θ and Φ as

$w(x,y,z,t) = 0 + \in \hat{w}(x,y,z,t), \hat{w} = (w_{00} + I^*\, w_{01}) e^{(ikx+\sigma t)}$

$\theta(x,y,z,t) = \theta_b + \in \hat{\theta}\,(x,y,z,t),\ \hat{\theta} = (\theta_{00} + I^*\, \theta_{01}) e^{(ikx+\sigma t)}$

$\Phi(x,y,z,t) = \Phi_b + \in \hat{\phi}\ (x,y,z,t)\ ,\ \hat{\Phi} = (\Phi_{00} + I^*\, \Phi_{01}) e^{(ikx+\sigma t)}$

$R = R_{00} + I^*\, R_{01}$ (23)

Where ϵ perturbation parameter with $\in << 1$ and $\hat{w}$, $\hat{\theta}$ and $\hat{\varphi}$ are the perturbed quantities, which are expanded in terms of a small parameter I^*. Here k is the horizontal component of the wave number α and σ is the growth rate of the disturbance. Further W_{00}, W_{01}, θ_{00}, θ_{01}, Φ_{00}, Φ_{01} are purely functions of z. The basic state analysis has been performed in sec 4.

Next from equations (6) to (9) and (23), the localized perturbed system is given by

$\vec{q} + \nabla\, \hat{p}\ + R\, \hat{\theta}\hat{k} + \frac{I^*}{1-\varphi_B}[(\partial_t - \partial_z)]\, \vec{q}\ = 0$ (24)

$(\partial_t - \partial_z - \nabla^2)\ \hat{\theta}\ - S(\partial_t - \partial_z)\ \hat{\phi} + \hat{W} D\theta_b = 0$ (25)

$(\partial_t - \partial_z)\ [(\theta_b\ - C)\ \Phi - (1 - \Phi_b)\ \hat{\theta}\ ] - \hat{W} D\theta_b = 0$ (26)

Together with the boundary conditions (27) to (28)

The upper boundary i.e., mush-liquid interface is assumed to be flat and impermeable with zero solid fraction and whose temperature is equivalent to the liquidus temperature of the mushy layer as discussed earlier. Further the continuity of heat flux at the boundary is also assumed in order to facilitate the solution process. Mathematically, these are expressed as

$\hat{w} = \hat{\theta} = 0$ @ $z = 0$ (27)

$\hat{w} = \hat{\theta} = \hat{\varphi} = 0$ @ $z = \delta$ (28)

All the quantities have their predefined meanings (In all the future expressions, the 'caps' are dropped for the sake of simplicity). Next, we eliminate the pressure in (24),by applying curl twice and then consider the z – component of the equation for further analysis. In fact, applying the transformation

$\frac{\partial}{\partial z}[\frac{\partial(ith)}{\partial x} + \frac{\partial(jth)}{\partial y}] - \nabla_1^{\ 2}(kth)$ (29)

on the momentum equation (in the component form) is same as that of applying curl twice and consider the z-component of the result. Now by using the following result, the resulting equations are obtained:

$\nabla x \nabla x(\frac{1}{(1-\emptyset_b)}[\ \partial_t - \partial_z]\ \hat{q}\ ) = -\frac{1}{(1-\emptyset_b)}\ [(\partial_t - \partial_z)\ \nabla^2 w + \frac{D\emptyset_b}{(1-\emptyset_b)}\ (\partial_t - \partial_z) Dw]$ (29a)

where $D = \frac{d}{dz}$ .We write the resulting perturbed system as

$\nabla^2 w + R\ \nabla_1^{\ 2}\theta + \frac{I^*}{(1-\emptyset_b)}\ [\frac{D\emptyset b}{(1-\emptyset_b)}\ (\partial_t - \partial_z)Dw\ + (\partial_t - \partial_z)\nabla^2 w\ ] = 0$ (30)

$(\partial_t - \partial_z + \nabla^2)\ \theta - S((\partial_t - \partial_z)\emptyset - (1-\emptyset b)\ \theta\ ] + D\emptyset b w = 0$ (31)

Now by using the expansion (23) through the normal mode approach for the physical variables, we write the system of order $(I^*)^{\ 0}$ as

$(D^2 - \alpha^2)\ W_{00} - R_{00}\ \alpha^2\theta_{00} = 0$ (32)

$(D^2 + D - \alpha^2 - \sigma)\theta_{00} - S\ (D - \sigma)\ \emptyset_{00} - D\theta_b\ W_{00} = 0$ (33)

$(D - \sigma)\ [(\theta_b - C)\ \emptyset_{00} - (1 - \emptyset_b)\theta_{00}\ ] + D\theta_b\ W_{00} = 0$ (34)

The above system can be expressed as

$L\ \alpha_{00} = 0$

where $\alpha_{00} = [w_{00}, \theta_{00}, \emptyset_{00}]^{\ T}$, T denotes the transpose and $\mathcal{L}$ is the linear operator given by

$$\mathcal{L} = \begin{pmatrix} (D^2 - \alpha^2) & -R_{00}\ \alpha^2 & 0 \\ -D\ \theta_b & D^2 + D - \alpha^2 - \sigma & -S\ (D - \sigma) \\ D\ \theta_b & (D - \sigma)(\theta_b - C) + D\theta_b & -(D - \sigma)(1 - \phi_b) + D\phi_b \end{pmatrix} \tag{35}$$

By letting $\Theta_{00} = -\text{Sin}\pi z$, The solutions for the system (32) to (34) are given by

$\emptyset_{00} = A_1^*\ \text{Sin}\pi z + A_2^*\ \text{Cos}\pi z + b_1(z)$ (36)

$w_{00} = A_1\text{Sin}\pi z + A_2\text{Cos}\pi z + b_2(z)$

where

$b_1(z) = A_2^* Z$ and $b_2(z) = A_2\ (2Z - 1)$

$A_1^* = \frac{-1}{S}$,

$A_2^* = \frac{(\pi^2 + \alpha^2)\ A_{1*}}{\pi C_3}$

$A_1 = \frac{C_1}{C_2}(\pi\ A_2^*(C+1))$

$A_2 = \frac{A1^*}{C_2}(\pi + 1)(C+1)$

For the marginal stability $\sigma = 0$ and by using the above results in (32), the expression for R_{00} is obtained:

$R_{00} = \frac{(\pi^2 + \alpha^2)\ A_1}{\alpha^2}$ (37)

whereα is the wave number. Now in order to compute $R = R_{00} + I^*\ R_{01}$,we consider from (6) to (9) and (23), the system of order I* as

$(D^2-\alpha^2)W_{01} - R_{00}\,\alpha^2\theta_{01} = R_{01}\,\alpha^2\theta_{00} - \frac{1}{(1-\emptyset_b)}\left[\frac{D\emptyset_b}{(1-\emptyset_b)}(-D)Dw_{00} + (-D)\nabla^2 w_{00}\right]$ (38)

$(D^2 + D - \alpha^2 - \sigma)\,\theta_{01} - S(D-\sigma)\,\emptyset_{01} - D\theta_b\,W_{01} = 0$ (39)

$(D-\sigma)\,[(\theta_b - C)\,\emptyset_{01} - (1-\emptyset_b)\theta_{01}] + D\,\theta_b\,W_{01} = 0$ (40)

The above system in the matrix form is given by

$T\alpha_{01} = f(\alpha_{00})$ (41)

where

$\alpha_{01} = [w_{01}, \theta_{01}, \emptyset_{01}]^T$

By using the results of the zeroth- order system and the solvability condition, the inhomogeneous equation (38) is solved for R_{01} in which the inertial effects appear. Thus we have

$R_{01} = \frac{\pi^2 C_4}{\alpha^2 A_1 P_1^2}[(1+P_2)(A_1^2 + A_2^2) - \frac{2}{\pi} P_2 A_1 A_2] + \frac{\pi.P_2}{2\,\alpha^2 A_1 P_1}[(\pi^2 + \alpha^2)(A_1^2 - A_2^2)]$ (42)

where

$C_4 = C_2 A_1^*, P_1 = (1 - C_3), P_2 = \frac{C_4}{P_1}$

The other quantities have their predefined meanings. The critical value α_c corresponding to R_{00} is π and $R_{0C} = 2A_1$. Marginal stability curves for $R = R_{00} + I^* R_{01}$ are presented in fig.2 for the experimental values of the parameters S= 3.2, C= 9, θ_∞ = 0.6, 0.7, 0.8 and I* = 0.0, 0.05, 0.1.etc respectively. The results are in excellent agreement with the numerical results of [25]. In order to study the effects of inertia on the vertical component of velocity w, temperature θ and the solid fraction ∅, the first order system O(I*) is solved by using the results of the basic state and the zeroth order system, with the computed value of R_{01} by solving the respective differential equations along with the boundary conditions. The following results are obtained:

$w_{01} = C_{10}\text{Sin}\pi z + C_{11}\text{Cos}\pi z + b_3(z)$ (43)

$\Theta_{01} = C_{12}\text{Sin}\pi z + C_{13}\text{Cos}\pi z + b_4(z)$ (44)

$\Phi_{01} = C_{15}\text{Sin}\pi z + C_{16}\text{Cos}\pi z + b_5(z)$ (45)

where

$C_5 = -\alpha^2 R_{01} - \frac{\pi^2 A_1 C_4 S^2}{(C+1)^2} - \frac{\pi.(\pi^2 + \alpha^2)}{(C+1)}(SA_2)$

$C_6 = -[\frac{\pi^2 A_2 C_4 S^2}{(C+1)^2} + \frac{\pi.(\pi^2 + \alpha^2)}{(C+1)}](SA_1)$

$C_7 = -(\pi^2 + \alpha^2)C_5 - \pi(C+2)C_6$

$C_8 = -(\pi^2 + \alpha^2)C_6 + \pi(C+2)C_5$

$C_9 = \frac{\alpha^2 R_{00} C_2 (C+S+1)}{C+1}$

$C_{10} = [\frac{(\pi^2 + \alpha^2)C_7(1+C_9) - (\pi(C+2)*C_8)(\pi^2+\alpha^2)}{(\pi^2+\alpha^2)^2(1+C_9)^2 + \pi^2 (C+2)^2}]$

$C_{11} = [\frac{(\pi^2 + \alpha^2)C_8(1+C_9) + (\pi(C+2)C_7)(\pi^2+\alpha^2)(\pi^2+\alpha^2)}{(\pi^2+\alpha^2)^2(1+C_9)^2 + \pi^2 (C+2)^2}]$

$C_{12} = [\frac{(\pi^2 + \alpha^2)C_{10} - C_5}{\alpha^2 R_{00}}]$

$C_{13} = [\frac{(\pi^2 + \alpha^2)C_{11} - C_6}{\alpha^2 R_{00}}]$

$C_{15} = \frac{-(C+1)}{\pi}[C_{11} + C_{13}(1 - C_3)]$

$C_{16} = \frac{[(C+1)[C_{10} + C_{12}(1-C_3)]}{\pi}$

$b_3(Z) = (2Z-1)C_{11}$, $b_4(Z) = (2Z-1)C_{13}$

$b_5 = C_{16}(Z)$

By using the above results the values of

$w = w_{00} + I^* w_{01}$, (46)

$\Theta = \Theta_{00} + I^*\Theta_{01}$ (47)

$\Phi = \Phi_{00} + I^*\,\Phi_{01}$ (48)

are computed and the profiles are presented in figs.**(2-5)** and **table 1**for the specified values of the parameters S, θ_∞, C and I*. The graphs clearly indicate

(i) the stabilizing effect of the inertial terms on the convection in a mushy layer with constant permeability.

(ii) the decrease in the non-uniformity in the w – profile (vertical velocity) for the increase in I * and w is maximum at the middle of the layer and

(iii) the increase in the non-uniformity in the profiles of the temperature as well as the solid fraction of the mushy layer for the increase in I *

VI. Results and discussion

As discussed earlier, the solution procedure consists of two stages. In the first stage, the basic state solutions is obtained by considering the steady motionless state and the corresponding solutions are functions of z only. These solutions are required for studying the linear stability analysis. In the second stage the perturbed system for zeroth and first order are considered along with the boundary conditions. To start with the solutions W_{00}, θ_{00} and Φ_{00} are found by solving the linear differential equations and then finally $R_{00}$and the critical wave number are determined. From the higher order system which is an inhomogeneous system, R_{01} is computed by applying the solvability conditions.

(i) By setting σ =0, marginal stability results are determined analytically for the experimental values of the parameters[26], Stefan number S concentration ratio C, far field temperature $\Theta_\infty$and the expansion parameter I*. After computing R_{00} the analytical expression for R_{01} is derived. From the zeroth order system, the expressions for W_{00}, θ_{00} and Φ_{00} are found analytically by using the basic state solutions. Then the Rayleigh number and the wave numbers are determined. The critical values of these numbers are computed. In the solution process, the far field temperature Θ_∞ is considered by the condition D θ_b $=\Theta_\infty$ @ z = δ (the top boundary).

(ii) From the first order system R_{01} is determined by performing suitable operations on the first order system of equations, since it is an inhomogeneous system. In fact the effect of inertia appears only in the first order system as a part of the inhomogeneous term. We then compute total R = R_{00} +I* $R_{01}$and present the graphs in figs (2 – 5) and table 1 for various specified values of the parameters S=3.2, C=9.0 and Θ_∞= 0.6, 0.7, 0.8 respectively. For the computation of total R the values of I* are taken as 0.0, 0.05 and 0.1 respectively. The results show that the marginal stability curves are extremely sensitive to the far-field temperature and the Concentration ratio. Also as Θ_∞ increases R decreases there by indicating the destabilizing nature of Θ_∞. The presence of inertia term increases the value of R as expected. Thus the chimney formation could be suppressed.

(iii) The solutions for W_{01}, θ_{01}, $\Phi_{01}$are obtained by solving the first order system of equations O(I*). The profiles of W= W_{00} + I* W_{01}, θ =θ_{00} + I* θ_{01}, Φ = Φ_{00}+I* Φ_{01} are drawn for I* =0.0, 0.05, 0.1 respectively.

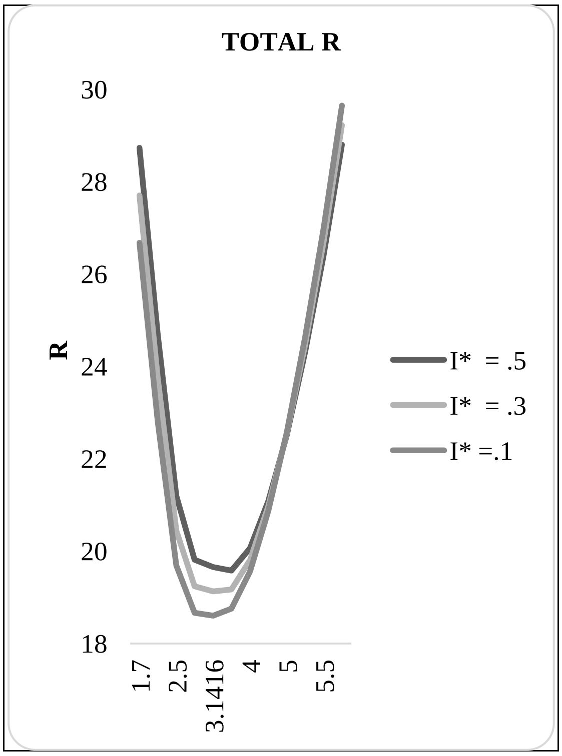


Fig.2 Total R vs α for I* = 0.5, 0.3 , 0.1

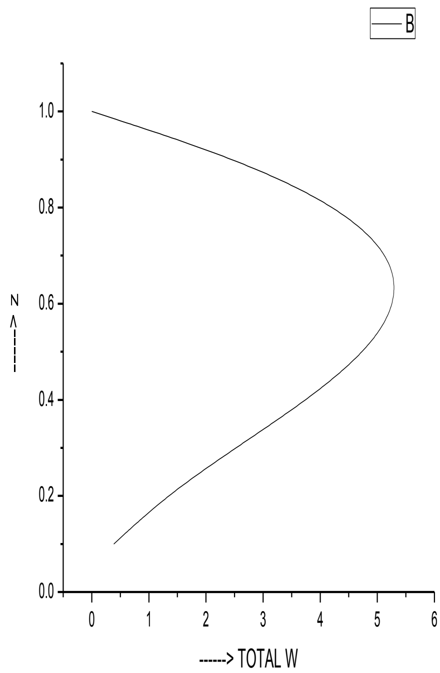
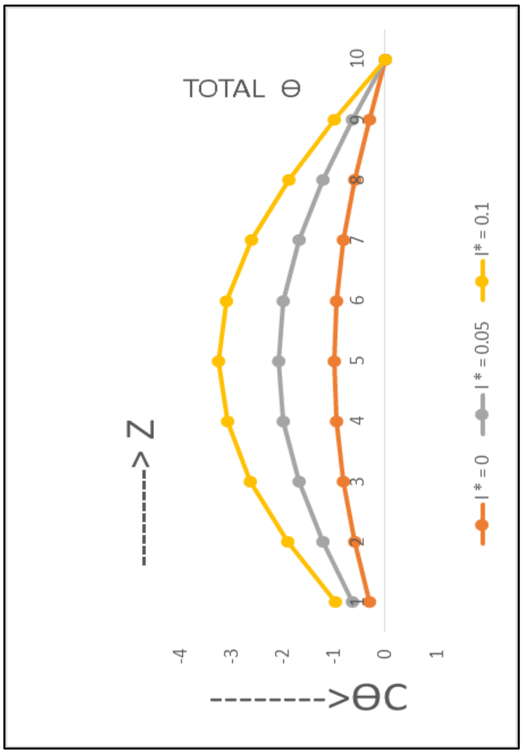


Fig. 3 Total W vs Z for I* = 0.05, 0.1 , 0



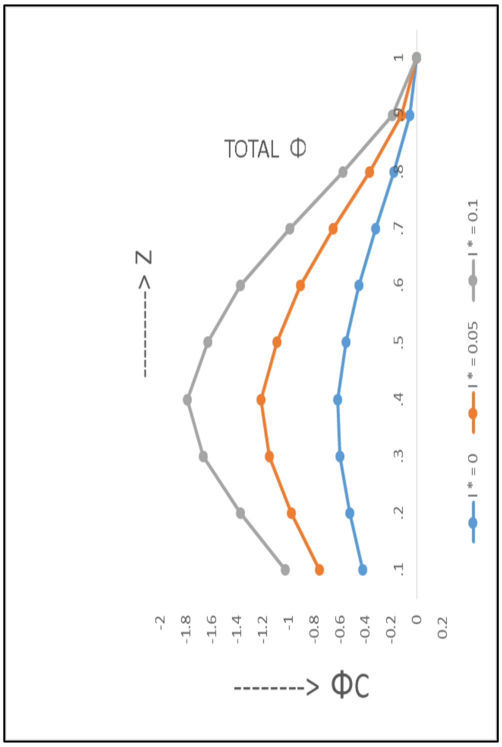


Fig. 4 Total Θ vs Z for I* = 0.05, 0.1 , 0

Fig. 5 Total Φ vs Z for I* = 0.05

	TOTAL $W = W_{00} + I^* W_{01}$		
Z	I* = 0	I * =0.05	I* = .1
0.1	0.384936766	0.632774898	0.137098633
0.2	1.3026524	1.825030792	0.780274008
0.3	2.520699251	3.304705401	1.736693102
0.4	3.777231021	4.771601769	2.782860273
0.5	4.80663422	5.926773768	3.686494671
0.6	5.365528389	6.501789165	4.229267612
0.7	5.256589985	6.285005762	4.228174208
0.8	4.347867875	5.142288502	3.553447247
0.9	2.585699687	3.030139606	2.141259768
1	-3.53115E-05	-4.35405E-05	-2.70825E-05

Table 1: The effect of I* on w

From fig.3,it is observed that the effect of increase in the inertia parameter I*, is to reduce the vertical component of the velocity W. Further, the non-uniformity of the velocity profile decreases as I* increases. Figs (3 - 5) indicate that the increase in the inertial effects increases the value of the temperature and solid fraction in the mushy zone. Accordingly the increase in the inertial effects increases the amount of non-uniformity in the temperature and the local solid fraction in the mushy layer by facilitating the suppression of chimney formation during the solidification process.

The following are the important conclusions drawn from the present investigation:

(i) Very sparse literature exists in the case of hydrodynamic convection in a mushy layer under different types of constraints. No analytical work is available with regard to mushy layer convection in the presence and absence of magnetic field. Therefore the present work is undertaken and the mathematical model is presented along with the necessary assumptions.

(ii) The method of analysis constitutes the application of modified perturbation technique. The determination of the Rayleigh number is done by setting $\sigma = 0$ and the results are in excellent agreement with the available numerical results in the limiting cases although, the analytical procedure is quite tedious, the results are accurate and well justified.

(iii) The method is found to be very elegant and effective in predicting the effect of the inertial terms on the convection in a mushy layer. The main objective of all research workers is to provide a suitable mathematical model which can predict the conditions under which the formation of freckles could be avoided during the solidification process of a binary alloy. This in turn facilitates the non-formation of chimneys that is formed due to the morphological instabilities at the interface of the mushy layer and the solid phase. Through the analytical approach it is shown that the complete solution of the basic zeroth and the first order systems could be accurately determined. This has not been undertaken in any of the available literature and hence the present work is first of its kind. Finally it is concluded that the introduction of the inertial effects suppresses the formation of freckles and increases the amount of non-uniformity present in the temperature and solid fraction of the mush- layer. Therefore through the proper choice of the governing parameters it is possible to have a complete control over the formation of chimneys during the solidification process of a binary alloy.

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