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ON FINITE ELEMENT ERROR ANALYSIS OF TWO - DIMENSIONAL TIME- INDEPENDENT PROBLEMS USING FEM

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***Abstract**: Poisson's equation is most common field governing equation to describe various physical natures. In this paper we focused our attention on the Two - dimensional time independent problems. The entire FEM including variational formulation, finite element formulation, choice of elements, choice of basis function,Isoparametric formulation, constructing stiff matrix, constructing load vector has been discussed. I conclude this research paper with some observations on finite element error analysis.*

***Key words**: FEM, Time independent problem, Possion equation, Finite element error analysis.*

I.Introduction

Basic idea of any numerical method for differential equation is to discretize the given continuous problem with large degree of freedom to obtain a discrete problem or system of equation with only finite unknown that may be solved using a computer. The Finite Element Method (FEM) is general technique for numerical solution of diffrential equations in science and engineering.The discretization process using a finite element method is diffrent from classical " Diffrence Method " where we replace derivatives by difference quotients[1]-[4]. In FEM we start from reformulation of given diffrential equation as an equivalent variational problem.

II.Proposed Algorithm

Here we consider the finite element formulation of 2-D possion equation [5]. Poisson's equation is most common field governing equation to describe various physical natures.

$$-\nabla(a.\nabla u) = f \qquad in \quad \Omega \tag{1}$$

Step[1]. Discretisation of Domain: In two dimensions, there is more then one simple geometric shape that can be used as finite element. The representation of a given region by a set of elements is called *mesh generation*.

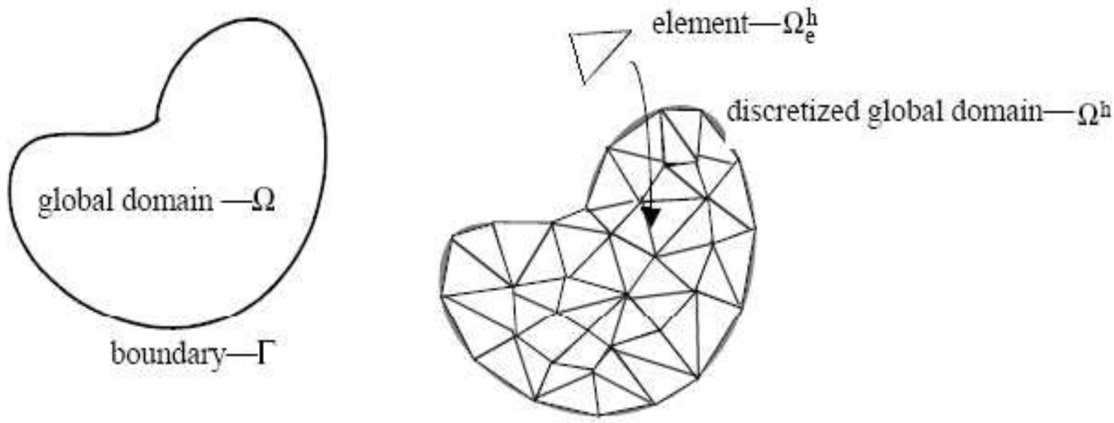

Figure (1)

Step[2]. Weak form: In the development of the weak form, we need only consider an arbitrary typical element. We assume Ω_e is such an element. Multiply (1) with a weight function w, which is assumed to be once differentiable with respect to x and y, and then integrate the resulting equation over Ω_e

$$\int_{\Omega_e} (-\nabla(a.\nabla u) - f)w.dx.dy \tag{2}$$

Using Green's theorem

$$\int_{\Omega_e} \left[a \frac{dw}{dx}\frac{du}{dx} - \frac{dw}{dy}\frac{du}{dy} - wf \right] dxdy - \int_{\Gamma^e} q_n w ds = 0 \tag{3}$$

where

$$q_n = n_x \frac{du}{dx} + n_y \frac{du}{dy}$$

Weak form (3) also called as variational problem forms the basis of the finite element model of (1).

Step[3].Finite Element Model: Suppose u is approximated over a typical finite element Ω_e by the expression

$$u(x,y) = U^e = \sum_{j=1}^{n} u_j^e \phi_j^e(x,y) \tag{4}$$

where u_j^e is the value of U^e at the j^{th} node (x_j, Y_j) of element, and ϕ_j^e are interpolation functions, with property $\phi_i^e(x_j, y_j) = \delta_{ij}$.

Substituting the finite element approximation (4) for u in to weak formulation(3)and n independent function for $w = \phi_1^e, \phi_2^e, \phi_3^e, \ldots\ldots, \phi_n^e$ we obtain n algebraic equation. The i^{th} algebraic equation is obtained by substituting $w = \phi_i^e$.

$$\sum_{j=1}^{n} \left\{ \int_{\Omega_e} a\left(\frac{d\phi_j^e}{dx}\frac{d\phi_i^e}{dx} - \frac{d\phi_j^e}{dy}\frac{d\phi_i^e}{dy} \right) dxdy \right\} u_j^e - \int_{\Gamma^e} q_n \phi_i^e ds - \int_{\Omega^e} f\phi_i^e dxdy = 0$$

$$where \quad i = 1,2,3,\ldots..n \tag{5}$$

or

$$\sum_{j=1}^{n} K_{ij}^e u_i^e = f_i^e + Q_i^e \tag{6}$$

In matrix notation, linear algebraic equations can be written as

$$\left[K^e\right]\left\{u^e\right\} = \left\{f^e\right\}\left\{Q^e\right\} \tag{7}$$

$\left[K^e\right]$ is called stiffness matrix.

Step[4]. Interpolation function for linear triangular element: One of the simplest two-dimensional elements is the three node triangular element. This is known as *linear triangular element*. It has three nodes at the vertices of the triangle and the variable interpolation within the element is linear in x and y, as

$$U^e = a + bx + cx^2 \tag{8}$$

contains three linear independent terms, and it is linear in both x and y. approximation must satisfy

$$U^e(x_i, y_i) = u_i^e \qquad (i = 1,2,3) \tag{9}$$

where $(i = 1,2,3)$ are global coordinates of the triangle . applying condition (9) on (8) we get

$$U^e = \sum_{j=1}^{n} u_j^e \phi_j^e(x,y)$$

where

$$\phi_1^e(x,y) = \frac{1}{2A_e}\left[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y\right]$$

$$\phi_2^e(x,y) = \frac{1}{2A_e}\left[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y\right]$$

$$\phi_3^e(x,y) = \frac{1}{2A_e}\left[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right] \quad (10)$$

A_e is the area of triangle Ω_e . Similarly one can obtain linear interpolating polynomial for linear rectangular element.

Step[5]. Isoparametric formulation: As we already discretise the domain in to finite number of triangular elements. In (5) we need to integrate the integrand over these triangular domains Ω_e . Since the orientation of triangle is not regular it is a difficult to find the limit of integration for these triangles which is different for each triangles. This problem may be overcome using concept of isoparametric formulation.

Figure (2) showing transformation of arbitrary triangle in to unit right angled triangle

The idea involves the transformation of a triangle to a unit right angled triangle by transformation of axis. so that integration can be carried out over unit right angled triangle (*Figure(2)*).

Step[6]. Evaluation of boundary integrals: Here we will consider the evaluation of boundary integrals of the type

$$Q_i^e = \int_{\Gamma^e} q_i^e \phi_i^e \, ds$$

where q_i^e is a known function of a distance s along the boundary Γ^e. It is not necessary to compute such integrals when a portion of Γ^e does not coincide with the boundary Γ of the domain Ω. On the partition of Γ^e that are in the interior of the domain Ω is cancelled by the q_i^e of q_i^e neighbouring element. This can be viewed as the equilibrium of the internal flux.

Step[7]. Assembly of element equation : The assembly of finite element equation is based on the same principal used in one- dimensional problems:

1. Continuity of the primary variables
2. " Equilibrium " of secondary variables

Global and element nodes are connected by a correspondence called connectivity relations which can be represented in terms of a matrix.

Step[8]. Imposition of boundary condition: Here we have to impose the boundary condition on the assembled set of algebraic equations.

III. Conclusion

The error introduced into the finite element solution of a given differential equation can be attributed to three basic sources:

(a) *Domain approximation error*, this is due to approximation of the domain. In one dimensional problem no approximation of domain is required as the domains are generally straight lines. In two dimensio-nal problems involving nonrectangular domains, domain approximation errors are introduced into the finite element solution.

(b) *Quadrature and finite arithmetic errors*, these are due to numerical evaluation of integrals, and round-off errors in the computation of numbers.

(c) *Approximation error*, this is due to approximation of solution u in an element .

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