Fixed Point Theorem in Complex Valued Metric Spaces for Continuity and Compatibility

Abstract: In this paper, we prove some common fixed point theorems in complex valued metric spaces for weakly compatible and continuity. Which generalized the well known result.

Keywords: Common fixed point, complex valued metric space, Weakly compatible mapping.

Mathematics Subject Classification: 47H10, 54H25.

I. Introduction

In 2011, Azam et al. [1] introduced the notion of complex valued metric space, which is a generalization of the classical metric space and established some fixed point result for mappings satisfying a rational inequality. In what follows, we recall some notations and definitions that will be utilized in our subsequent discussion. Let \( \mathbb{C} \) be the set of complex numbers and \( z_1, z_2 \in \mathbb{C} \): Define a partial order \( \preceq \) on \( \mathbb{C} \) as follows: \( z_1 \preceq z_2 \) if and only if \( \Re(z_1) \leq \Re(z_2) \), \( \Im(z_1) \leq \Im(z_2) \). Consequently, one can infer that \( z_1 \preceq z_2 \) if one of the following conditions is satisfied: (i) \( \Re(z_1) = \Re(z_2) \), \( \Im(z_1) \leq \Im(z_2) \) (ii) \( \Re(z_1) < \Re(z_2) \), \( \Im(z_1) = \Im(z_2) \) (iii) \( \Re(z_1) < \Re(z_2) \), \( \Im(z_1) < \Im(z_2) \) (iv) \( \Re(z_1) = \Re(z_2) \), \( \Im(z_1) = \Im(z_2) \) In particular, we write \( z_1 \preceq z_2 \) if \( z_1 \neq z_2 \) and one of (i), (ii), and (iii) is satisfied and we write \( z_1 \preceq z_2 \) if only (iii) is satisfied. Notice that \( 0 \preceq z_1 \preceq z_2 \Rightarrow |z_1| \leq |z_2| \) and \( z_1 \preceq z_2, z_2 \preceq z_3 \Rightarrow z_1 \preceq z_3 \). Recently, Azam et al. [1] introduced the notion of complex valued metric spaces and established some fixed point results for a pair of mappings for contraction conditions satisfying a rational expression. Though complex valued metric spaces form a special class of cone metric space, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many results of analysis cannot be generalized to cone metric spaces.

II. PRELIMINARIES

Definition 2.1. [4] Let \( X \) be a nonempty set whereas \( \mathbb{C} \) be the set of complex numbers. Suppose that the mapping \( d : X \times X \to \mathbb{C} \) satisfies following conditions:
1. (1) \( 0 \preceq d(x, y) \) for all \( x, y \in X \) and \( d(x, y) = 0 \) if and only if \( x = y \);
2. (2) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);
3. (3) \( d(x, y) \preceq d(x, z) + d(z, y) \) for all \( x, y, z \in X \).

Then \( d \) is called a complex valued metric on \( X \), and \( (X, d) \) is called a complex valued metric space.

Definition 2.2. [8] Let \( (X, d) \) be a complex valued metric space and \( \{x_n\} \) be a sequence in \( X \).
1. (1) If for every \( c \in \mathbb{C} \) with \( 0 < c \), there exists \( N \in \mathbb{N} \) such that \( d(x_n, x) < c \) for all \( n \geq N \) then \( \{x_n\} \) is said to be convergent to \( x \in X \), and we denote this by \( x_n \to x \) as \( n \to \infty \) or \( \lim_{n \to \infty} x_n = x \).
2. (2) If for every \( c \in \mathbb{C} \) with \( 0 < c \), there exists \( N \in \mathbb{N} \) such that \( d(x_n, x_{n+m}) < c \) for all \( n \geq N \), where \( m \in \mathbb{N} \), then \( \{x_n\} \) is said to be Cauchy sequence.
3. (3) If every Cauchy sequence in \( X \) is convergent, then \( (X, d) \) is said to be a complete complex valued metric space.

Lemma 2.3. [8] Let \( (X, d) \) be a complex valued metric space and \( \{x_n\} \) be a sequence in \( X \). Then \( \{x_n\} \) converges to \( x \) if and only if \( d(x_n, x) \to 0 \) as \( n \to \infty \).

Lemma 2.4. [8] Let \( (X, d) \) be a complex valued metric space and \( \{x_n\} \) be a sequence in \( X \). Then \( \{x_n\} \) is a Cauchy sequence if and only if \( d(x_n, x_m) \to 0 \) as \( n \to \infty \) where \( m \in \mathbb{N} \).

Definition 2.5. [8] Let \( f \) and \( g \) be two self-mappings of a metric space \( (X, d) \). Then a pair \((f, g)\) is said to be weakly compatible if they commute at coincidence points.

Definition 2.6. [3] Let A and S be mappings from a complete metric space \( X \) into itself. The mappings A and S are said to be compatible if \( \lim_{n \to \infty} d(Ax_n, Sx_n) = 0 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x \) for some \( x \in X \).

III. Main Result

Theorem 3.1: Let \( (X, d) \) be a complex valued metric space and \( A, B, D, M, S \) and \( T \) be six self mappings in \( X \) satisfying the condition:
1. \( S(X) \subseteq BD(X) \) and \( T(X) \subseteq AM(X) \)
2. For each \( x, y \in X \), such that \( x \neq y \), \( d(Ty, AMx) + d(Sx, BDy) + d(BDy, AMx) \neq 0 \), where \( \alpha, \beta, \gamma \) and \( \eta \) are non negative real number with \( \alpha + \beta + 2\gamma + \eta < 1 \), or \( d(Sx, Ty) = 0 \) if \( d(Ty, AMx) + d(Sx, BDy) + d(BDy, AMx) = 0 \), such that
\[
d(Sx, Ty) \leq \alpha \left[ d(AMx, Sx) + d(BDy, Ty) + d(AMx, Sx) \right] + \beta \max \{ d(AMx, BDy), d(AMx, Sx), d(BDy, Sx) \} + \\
\gamma \{ d(BDy, Ty) + d(Ty, AMx) + d(Sx, BDy) \} + \eta \left[ d(Ty, AMx) + d(Sx, BDy) + d(BDy, AMx) \right]
\]
3. The pair \((AM, S)\) and \((BD, T)\) are commute.
4. The pair \((AM, S)\) and \((BD, T)\) are weakly compatible.

**Proof:** Let \( x_0 \in X \). Since \( S(X) \subseteq BD(X) \) and \( T(X) \subseteq AM(X) \), define for each \( n \geq 0 \), the sequence \( \{y_n\} \) in \( X \) by
\[
y_{2n+1} = Sx_{2n} = BDx_{2n+1} \quad \text{and} \quad y_{2n+2} = Tx_{2n+1} = AMx_{2n+2} \quad n = 0, 1, 2, \ldots
\]
Then
\[
d(y_{2n+1}, y_{2n+2}) = d(Sx_{2n}, Tx_{2n+1}) \leq \alpha \left[ d(AMx_{2n}, Sx_{2n}) + d(BDx_{2n+1}, Tx_{2n+1}) + d(AMx_{2n}, Sx_{2n}) \right] + \\
\beta \max \{ d(AMx_{2n}, BDx_{2n+1}), d(AMx_{2n}, Sx_{2n}), d(BDx_{2n+1}, Sx_{2n}) \} + \\
\gamma \left[ d(BDx_{2n+1}, Tx_{2n+1}) + d(Tx_{2n+1}, AMx_{2n+2}) + d(Sx_{2n}, BDx_{2n+1}) \right] + \\
\eta \left[ d(Tx_{2n+1}, BDx_{2n+1}) + d(Sx_{2n}, AMx_{2n+2}) + d(BDx_{2n+1}, AMx_{2n}) \right]
\]
\[
d(y_{2n+1}, y_{2n+2}) \leq \alpha \left[ d(AMx_{2n+1}, Sx_{2n}) + d(BDx_{2n+2}, Tx_{2n+1}) + d(AMx_{2n+2}, Sx_{2n+1}) \right] + \\
\beta \max \{ d(AMx_{2n+1}, BDx_{2n+2}), d(AMx_{2n+2}, Sx_{2n}), d(BDx_{2n+2}, AMx_{2n+2}) \} + \\
\gamma \left[ d(BDx_{2n+2}, Tx_{2n+1}) + d(Tx_{2n+1}, AMx_{2n+2}) + d(Sx_{2n+1}, BDx_{2n+2}) \right] + \\
\eta \left[ d(Tx_{2n+1}, BDx_{2n+1}) + d(Sx_{2n+1}, AMx_{2n+2}) + d(BDx_{2n+1}, AMx_{2n}) \right]
\]
\[
d(y_{2n+3}, y_{2n+4}) \leq \alpha \left[ d(AMx_{2n+2}, Sx_{2n}) + d(BDx_{2n+3}, Tx_{2n+1}) + d(AMx_{2n+3}, Sx_{2n+1}) \right] + \\
\beta \max \{ d(AMx_{2n+2}, BDx_{2n+3}), d(AMx_{2n+3}, Sx_{2n}), d(BDx_{2n+3}, AMx_{2n+3}) \} + \\
\gamma \left[ d(BDx_{2n+3}, Tx_{2n+1}) + d(Tx_{2n+1}, AMx_{2n+2}) + d(Sx_{2n+1}, BDx_{2n+1}) \right] + \\
\eta \left[ d(Tx_{2n+1}, BDx_{2n+1}) + d(Sx_{2n+1}, AMx_{2n}) + d(BDx_{2n+1}, AMx_{2n}) \right]
\]
That is
\[
|d(y_{2n+3}, y_{2n+4})| \leq \alpha (|d(y_{2n+1}, y_{2n+2})| + |d(y_{2n+2}, y_{2n+3})| + |d(y_{2n+3}, y_{2n+4})|) \cdot (3.1)
\]
Similarly
\[
d(y_{2n+3}, y_{2n+4}) \leq \alpha \left[ d(AMx_{2n+2}, Sx_{2n+1}) + d(BDx_{2n+3}, Tx_{2n+1}) + d(AMx_{2n+3}, Sx_{2n+2}) \right] + \\
\beta \max \{ d(AMx_{2n+2}, BDx_{2n+3}), d(AMx_{2n+3}, Sx_{2n+1}), d(BDx_{2n+3}, AMx_{2n+2}) \} + \\
\gamma \left[ d(BDx_{2n+3}, Tx_{2n+1}) + d(Tx_{2n+1}, AMx_{2n+2}) + d(Sx_{2n+2}, BDx_{2n+1}) \right] + \\
\eta \left[ d(Tx_{2n+1}, BDx_{2n+1}) + d(Sx_{2n+2}, AMx_{2n}) + d(BDx_{2n+1}, AMx_{2n}) \right]
\]
\[
d(y_{2n+3}, y_{2n+4}) \leq \alpha \left[ d(AMx_{2n+2}, Sx_{2n+1}) + d(BDx_{2n+3}, Tx_{2n+1}) + d(AMx_{2n+3}, Sx_{2n+2}) \right] + \\
\beta \max \{ d(AMx_{2n+2}, BDx_{2n+3}), d(AMx_{2n+3}, Sx_{2n+1}), d(BDx_{2n+3}, AMx_{2n+2}) \} + \\
\gamma \left[ d(BDx_{2n+3}, Tx_{2n+1}) + d(Tx_{2n+1}, AMx_{2n+2}) + d(Sx_{2n+2}, BDx_{2n+1}) \right] + \\
\eta \left[ d(Tx_{2n+1}, BDx_{2n+1}) + d(Sx_{2n+2}, AMx_{2n}) + d(BDx_{2n+1}, AMx_{2n}) \right]
\]
That is
\[
|d(y_{2n+3}, y_{2n+4})| \leq \alpha (|d(y_{2n+1}, y_{2n+2})| + |d(y_{2n+2}, y_{2n+3})| + |d(y_{2n+3}, y_{2n+4})|) \cdot (3.2)
\]
Therefore form (3.1) and (3.2)
\[
d(y_{2n+1}, y_{2n+2}) \leq \alpha (|d(y_{2n+1}, y_{2n+2})| + |d(y_{2n+2}, y_{2n+3})| + |d(y_{2n+3}, y_{2n+4})|) \]
That is \(\sum_{m=0}^{n} |d(y_{2n+1}, y_{2n+2})| \leq \frac{1}{1-\alpha} |d(y_{0}, y_{1})| \)

Hence \(d(y_{m}, y_{n}) \leq \frac{1}{1-\alpha} |d(y_{0}, y_{1})| \rightarrow 0 \) as \( m, n \rightarrow \infty \). That is \( lim_{m \rightarrow \infty} d(y_{m}, y_{n}) = 0 \). Hence \(\{y_{n}\}\) is a Cauchy sequence. Since \( X \) is complete, so \(\{y_{n}\}\) converges to some point \( z \). Therefore
\[
d(y_{m}, y_{n}) \leq d(y_{m}, y_{n}) + d(y_{n}, y_{m}) \rightarrow 0 \quad \text{as} \quad m, n \rightarrow \infty \]

Hence \(\{y_{n}\}\) converges to some point \( z \). Since \( AMz = BDz = Tu = \{0, 1, 2, \ldots \} \), then \( AMz \) and \( BDz \) are weakly compatible. Therefore
\[
d(z, AMz) = d(z, BDz) = 0 \]

Hence \( z \) is a coincidence point. Hence \( Sz = S(AMz) = AM(Sz) = AMz \). And \( BDz = BD(Tu) = Tu = z \). Since the pair \((AM, S)\) and \((BD, T)\) are weakly compatible. Then then commute at their coincidence point. Hence \( Sz = S(AMz) = AM(Sz) = AMz \). And \( BDz = BD(Tu) = Tu = z \).
Case (I): Now, we shall show that $Tz = Sz$, form (2) putting $x = z$ and $y = x_{2n+1}$ we have

$$d(Sz, Tx_{2n+1}) \leq \alpha \left[ d(Amz, Sz) + d(BDx_{2n+1}, Tz_{2n+1})d(Amz, Sz) \right]$$

$$\quad \quad \quad + \beta \max\{d(Amz, BDx_{2n+1}), d(Amz, Sz), d(BDx_{2n+1}, Sz)\}$$

$$\quad \quad \quad + \gamma \left( d(BDx_{2n+1}, Tz_{2n+1}) + d(Tz_{2n+1}, Amz) + d(Sz, BDx_{2n+1}) \right)$$

$$\quad \quad \quad + \eta \left[ d(Tz_{2n+1}, Amz) + d(Sz, BDx_{2n+1}) + d(BDx_{2n+1}, Amz) \right]$$

$$d(Sz, y_{2n+2}) \leq \alpha \left[ d(Sz, Sz) + d(y_{2n+1}, y_{2n+2})d(Sz, Sz) \right]$$

$$\quad \quad \quad + \beta \max\{d(Sz, y_{2n+1}), d(Sz, Sz), d(y_{2n+1}, Sz)\}$$

$$\quad \quad \quad + \gamma \left( d(y_{2n+1}, y_{2n+2}) + d(Sz, y_{2n+2}) + d(Sz, y_{2n+1}) \right) + \eta \left[ d(y_{2n+2}, y_{2n+1})d(Sz, Sz) \right]$$

Letting $n \to \infty$, we get

$$d(Sz, z) \leq \alpha \left[ d(Sz, Sz) + d(z, z)d(Sz, Sz) \right] + \beta \max\{d(Sz, z), d(Sz, Sz), d(z, Sz)\}$$

$$\quad \quad \quad + \gamma \left( d(z, z) + d(z, Sz) + d(Sz, z) \right) + \eta \left[ d(Sz, Sz) + d(z, z) + d(z, Sz) \right]$$

$$d(Sz, z) \leq \beta d(Sz, z) + 2\gamma d(Sz, z)$$

Then $d(Sz, z) \leq (\beta + 2\gamma)|d(Sz, z)|$ That is, $|d(Sz, z)| \leq (\beta + 2\gamma)|d(Sz, z)|$

Which is contradiction $\beta + 2\gamma < 1$. Therefore $Sz = z$, since $AMz = Sz$ which implies $AMz = z$. Now we prove that $Tz = z$. from (2), putting $x = y = z$, we get

$$d(Sz, Tz) \leq \alpha \left[ d(Amz, Sz) + d(BDx_{2n+1}, Tz_{2n+1})d(Amz, Sz) \right]$$

$$\quad \quad \quad + \beta \max\{d(Amz, BDx_{2n+1}), d(Amz, Sz), d(BDx_{2n+1}, Sz)\}$$

$$\quad \quad \quad + \gamma \left( d(BDx_{2n+1}, Tz_{2n+1}) + d(Tz_{2n+1}, Amz) + d(Sz, BDx_{2n+1}) \right)$$

$$\quad \quad \quad + \eta \left[ d(Tz_{2n+1}, Amz) + d(Sz, BDx_{2n+1}) + d(BDx_{2n+1}, Amz) \right]$$

$$d(Sz, Tz) \leq \alpha \left( d(Tz, Tz) + d(Tz, Amz) + d(Sz, BDz) \right) + \eta \left[ d(Tz, Amz) + d(Sz, BDz) + d(BDz, Amz) \right]$$

$$d(Tz, Tz) \leq \alpha \left[ d(Amz, Sz) + d(BDx_{2n+1}, Tz_{2n+1})d(Amz, Sz) \right]$$

$$\quad \quad \quad + \beta \max\{d(Amz, BDx_{2n+1}), d(Amz, Sz), d(BDx_{2n+1}, Sz)\}$$

$$\quad \quad \quad + \gamma \left( d(BDx_{2n+1}, Tz_{2n+1}) + d(Tz_{2n+1}, Amz) + d(Sz, BDx_{2n+1}) \right)$$

$$\quad \quad \quad + \eta \left[ d(Tz_{2n+1}, Amz) + d(Sz, BDx_{2n+1}) + d(BDx_{2n+1}, Amz) \right]$$

$$d(Sz, Tz) \leq \alpha \left[ d(Sz, Sz) + d(z, z)d(Sz, Sz) \right]$$

$$\quad \quad \quad + \beta \max\{d(Sz, z), d(Sz, Sz), d(z, Sz)\}$$

$$\quad \quad \quad + \gamma \left( d(z, z) + d(z, Sz) + d(Sz, z) \right) + \eta \left[ d(Sz, Sz) + d(z, z) + d(z, Sz) \right]$$

$$d(Sz, z) \leq \beta d(Sz, z) + 2\gamma d(Sz, z)$$

Then $d(Sz, z) \leq (\beta + 2\gamma)|d(Tz, Tz)|$ That is, $|d(Sz, z)| \leq (\beta + 2\gamma)|d(Tz, Tz)|$

Which is contradiction $\beta + 2\gamma < 1$. Therefore $Tz = z$. since $BDz = Tz$ which implies $BDz = z$. Now we prove that $Mz = z$. from (2), putting $x = Mz$ and $y = z$, we get

$$d(Mz, Tz) \leq \alpha \left[ d(Am(Mz), Sz(Mz)) + d(BDz, Tz_{2n+1})d(Am(Mz), Sz(Mz)) \right]$$

$$\quad \quad \quad + \beta \max\{d(Am(Mz), BDz), d(Am(Mz), Sz(Mz)), d(BDz, Sz(Mz))\}$$

$$\quad \quad \quad + \gamma \left( d(BDz, Tz_{2n+1}) + d(Tz_{2n+1}, Amz(Mz)) + d(Sz, BDz, Sz(Mz)) \right)$$

$$\quad \quad \quad + \eta \left[ d(Tz_{2n+1}, Amz(Mz)) + d(Sz, BDz, Amz(Mz)) \right]$$

$$d(Mz, z) \leq \alpha \left[ d(Mz, Mz) + d(z, z)d(Mz, Mz) \right]$$

$$\quad \quad \quad + \beta \max\{d(Mz, Mz), d(Mz, Mz), d(z, Mz)\}$$

$$\quad \quad \quad + \gamma \left( d(z, z) + d(Mz, Mz) + d(Mz, z) \right) + \eta \left[ d(Mz, Mz) + d(z, Mz) + d(z, Mz) \right]$$

$$d(Mz, z) \leq \beta d(Mz, z) + 2\gamma d(Mz, z)$$

That is, $|d(Mz, z)| \leq (\beta + 2\gamma)|d(Mz, z)|$ Which is contradiction $\beta + 2\gamma < 1$. Therefore $Mz = z$. since $AMz = z$ which implies $Mz = z$. Now we prove that $Dz = z$. from (2), putting $x = z$ and $y = Dz$, we get

$$d(Sz, T(Dz)) \leq \alpha \left[ d(Amz, Sz) + d(BDz, T(Dz))d(Amz, Sz) \right]$$

$$\quad \quad \quad + \beta \max\{d(Amz, BDz), d(Amz, Sz), d(BDz, Sz)\}$$

$$\quad \quad \quad + \gamma \left( d(BDz, T(Dz)) + d(T(Dz), Amz) + d(Sz, BDz) \right)$$

$$\quad \quad \quad + \eta \left[ d(T(Dz), Amz) + d(Sz, BDz, Dz) + d(BDz, Amz) \right]$$

$$d(Dz, Dz) \leq \alpha \left[ d(z, z) + d(Dz, Dz)d(z, z) \right]$$

$$\quad \quad \quad + \beta \max\{d(z, Dz), d(z, z), d(Dz, Dz)\}$$

$$\quad \quad \quad + \gamma \left( d(Dz, Dz) + d(Dz, Dz) + d(z, Dz) \right) + \eta \left[ d(Dz, Dz)d(z, z) \right]$$
Then $d(z, Dz) \leq \beta d(z, Dz) + 2\gamma d(z, Dz)$. Therefore $d(z, Dz) \leq (\beta + 2\gamma) d(z, Dz)$, which is a contradiction. Hence $Dz = z$. Therefore $z$ is a common fixed point of $A, B, D, M, S$ and $T$.

**Uniqueness:** Let $u$ be another common fixed point of $A, B, D, M, S$ and $T$. Then, we have

$$d(Sz, Tu) \leq \alpha \left[ d(AMz, Sz) + d(BDz, Tu)d(AMz, Sz) \right] + \beta \max\{d(AMz, Bd), d(AMz, Sz), d(BDz, Sz)\} + \gamma \left[ (d(BDz, Tu) + d(Tu, Amz) + d(Sz, BDz)) \right] + \eta \left[ \frac{d(Tu, Bd)zd(Sz, Amz)}{d(Tu, Amz) + d(Sz, BdU) + d(BdU, Amz)} \right]$$

**Case II:** we consider the case: $d(Tx_{2n+1}, AMx_{2n}) + d(Sx_{2n}, BDx_{2n+1}) + d(BDx_{2n+1}, AMx_{2n}) = 0$ (for any $n$) implies that $d(Sx_{2n}, Tx_{2n+1}) = 0$. Thus we have $y_{2n+1} = x_{2n+1} = y_{2n} = x_{2n}$, such that $n_1 = m_1$ such that $n_1 = m_1 = m_2 = m_3$. Similarly, $y_{2n+2} = x_{2n+2}, y_{2n+3} = x_{2n+3}$. Therefore $z = u$. Hence $z$ is a unique common fixed point of $A, B, D, M, S$ and $T$.

**Uniqueness:** Let $v$ be another common fixed point of $A, B, D, M, S$ and $T$. Then, we have $v_1 = Sx_1 = Av_1 = Vx_1 = BVv_1 = BVv_1 = Vx_1$. Therefore $d(Tv_1, Amv_1) + d(Sn_1, Bdv_1) + d(Bdv_1, Amv_1) = 0$ so that $n_1 = m_1 = m_2 = m_3$. Hence $v_1$ is a unique common fixed point of $A, B, D, M, S$ and $T$.

**Corollary:** Let $(X, d)$ be a complex valued metric space and $A, B, D, M, S$ and $T$ be six self mappings in $X$ satisfying the condition:

1. $S(X) \subseteq D(X)$ and $T(X) \subseteq M(X)$
2. For each $x, y \in X$, such that $x \neq y$, $d(Ty, Mx) + d(Sx, Dy) + d(Dy, Mx) = 0$, where $\alpha, \beta, \gamma$ and $\eta$ are non negative real numbers with $\alpha + \beta + 2\gamma + \eta < 1$, or $d(Sx, Ty) = 0$ if $d(Ty, Mx) + d(Sx, Dy) + d(Dy, Mx) = 0$, such that

$$d(Sx, Ty) \leq \alpha \left[ d(Mx, Sx) + d(Dy, Ty)d(Mx, Sx) \right] + \beta \max\{d(Mx, Dy), d(Mx, Sx), d(Dy, Sx)\} + \gamma \left[ (d(Dy, Ty) + d(Ty, Mx) + d(Sx, Dy)) \right] + \eta \left[ \frac{d(Dy, Ty)d(Mx, Sx)}{d(Dy, Ty) + d(Sx, Dy) + d(Dy, Sx)} \right]$$

3. The pair $(M, S)$ and $(D, T)$ are weakly compatible.

Then $D, M, S$ and $T$ have a unique common fixed point.

**Corollary:** If $(M, S)$ and $(D, T)$ are four commuting self mappings defined on a complete complex valued metric space $(X, d)$ satisfying the condition:

$$d(S^nx, T^ny) \leq \alpha \left[ d(M^{nx}, S^{nx}) + d(D^{ny}, T^{ny})d(M^{nx}, S^{nx}) \right] + \beta \max\{d(M^{nx}, D^{ny}), d(M^{nx}, S^{nx}), d(D^{ny}, S^{nx})\} + \gamma \left[ d(D^{ny}, T^{ny}) + d(T^{ny}, M^{nx}) + d(S^{nx}, D^{ny}) \right] + \eta \left[ \frac{d(T^{ny}, M^{nx}) + d(S^{nx}, D^{ny}) + d(D^{ny}, M^{nx})}{d(T^{ny}, M^{nx}) + d(S^{nx}, D^{ny}) + d(D^{ny}, M^{nx})} \right]$$

For each $x, y \in X$, such that $x \neq y$, where $\alpha, \beta, \gamma$ and $\eta$ are non negative real numbers with $\alpha + \beta + 2\gamma + \eta < 1$, or $d(S^{nx}T^{ny}) = 0$ if $d(T^{ny}, M^{nx}) + d(S^{nx}, D^{ny}) + d(D^{ny}, M^{nx}) = 0$. Then $D, M, S$ and $T$ have a unique common fixed point.

**Theorem 3.2.** Let $A, B, D, M, S$ and $T$ be self mappings of a complete complex valued metric space $(X, d)$ satisfying conditions:

1. $S(X) \subseteq BD(X)$ and $T(X) \subseteq AM(X)$
2. For each $x, y \in X$, where $\alpha, \beta, \gamma$ and $\eta$ are non negative real numbers with $\alpha + \beta + 2\gamma + \eta < 1$, such that

$$d(Sx, Ty) \leq \alpha \left[ d(AMx, Sx) + d(BDy, Ty)d(AMx, Sx) \right] + \beta \max\{d(AMx, Bd), d(AMx, Sx), d(BDy, Sx)\} + \gamma \left[ d(BDy, Ty) + d(Ty, AMx) + d(Sx, BdY) \right] + \eta \left[ \frac{d(Ty, BdY)d(Sx, AmX)}{d(Ty, AMx) + d(Sx, BdY) + d(BdY, AMx)} \right]$$
3. (AM, S) are compatible, and AM or S is continuous and (BD, T) are weakly compatible.

4. (BD, T) are compatible, and BD or T is continuous and (AM, S) are weakly compatible.

Then A, B, D, M, S and T have a unique common fixed point.

**Proof:** By above theorem \( \{y_n\} \) is a Cauchy sequence. Since \( X \) is completed, so \( \{y_n\} \) converges to some point \( z \). Thus subsequence \( \{Sx_{2n}\} \), \( \{Bdx_{2n+1}\} \) and \( \{AMx_{2n+2}\} \) also converges to \( z \). That is \( \lim_{n \to \infty} y_n = \lim_{n \to \infty} Sx_{2n} = \lim_{n \to \infty} Bdx_{2n+1} = \lim_{n \to \infty} AMx_{2n+2} = \lim_{n \to \infty} Tx_{2n+1} = z \) (3.3)

Assume that \( S \) is continuous. Since (AM, S) are compatible, we have

\[ \lim_{n \to \infty} AM(Sx_{2n+2}) = \lim_{n \to \infty} S(AMx_{2n+2}) = Sz \] (3.4)

Now putting \( x = x_{2n+2}, y = x_{2n+1} \) then we have

\[
d(AM(Sx_{2n+2}), Tx_{2n+1}) \leq \alpha \frac{d(AMx_{2n+2}, Sx_{2n+2}) + d(Bdx_{2n+1}, Tx_{2n+1})d(AMx_{2n+2}, Sx_{2n+2})}{1 + d(Sx_{2n+2}, Tx_{2n+1})} + \beta \max\{d(AMx_{2n+2}, Bdx_{2n+1}), d(AMx_{2n+2}, Sx_{2n+2}), d(Bdx_{2n+1}, Sx_{2n+2})\}
\]

Adding \( \gamma d(Bdx_{2n+1}, AMx_{2n+2}) + \eta d(Bdx_{2n+1}, Bdx_{2n+1})d(Sx_{2n+2}, AMx_{2n+2}) \)

Letting \( n \to \infty \), in the above inequality and using (3.3) and (3.4), we get

\[
d(Sz, z) \leq \alpha \left( \frac{d(z, z) + d(z, z)d(z, z)}{1 + d(z, z)} \right) + \beta \max\{d(z, z), d(z, z), d(z, z)\}
\]

\[
d(Sz, z) \leq 0 \text{ that is } |d(Sz, z)| \leq 0 \text{ hence } Sz = z. \text{ Now putting } x = z \text{ and } y = x_{2n+1} \text{ in (2) we have}
\]

\[
d(Sz, Tx_{2n+1}) \leq \alpha \left( \frac{d(Amz, z) + d(z, z)d(Amz, z)}{1 + d(Sz, Tx_{2n+1})} \right) + \beta \max\{d(z, z), d(z, z), d(z, z)\}
\]

\[
d(Sz, z) \leq \alpha \left( \frac{d(Amz, z) + d(z, z)d(Amz, z)}{1 + d(Sz, z)} \right) + \beta \max\{d(z, z), d(z, z), d(z, z)\} + \gamma \left( d(z, z) + d(z, z) + d(z, z) \right) + \eta \frac{d(z, z)d(z, z)}{d(z, z) + d(z, z) + d(z, z)}
\]

Letting \( n \to \infty \), we have

\[
d(z, z) \leq \alpha \frac{d(Amz, z) + d(z, z)d(Amz, z)}{1 + d(Sz, Tw)} + \beta \max\{d(Amz, Tw), d(Amz, Sz), d(BdTw, Sw)\}
\]

\[
d(Sz, Tw) \leq \alpha \left( \frac{d(Amz, Sz) + d(BdTw, Tz) d(Amz, Sz)}{1 + d(Sz, Tw)} + \gamma \left( d(BdTw, Tw) + d(Tw, Amz) + d(Sz, BdTw) \right) \right)
\]

\[
d(z, Tw) \leq \alpha \left( \frac{d(z, z) + d(z, Tw) d(z, z)}{1 + d(Sz, Tw)} + \gamma \left( d(Tw, z) + d(Tw, z) + d(z, z) \right) \right)
\]

Which is contradiction to \( 2 \gamma < 1 \). Therefore \( Tw = z \), hence \( BDw = z = Tw \). Thus \( BDw = Tw \). Since \( B \) and \( T \) are weakly compatible then \( BDz = BD(Tw) = T(BDw) = Tz \). Thus \( z \) is a coincidence point of \( BD \) and \( T \) now to prove \( Tw = z \), from (2) putting \( x = z \) and \( y = z \).
That is, \( \|d(Tz, z)\| \geq (\beta + 2\gamma)d(Tz, z) \). Which is contradiction \( \beta + 2\gamma < 1 \).

Therefore \( Tz = z \). since \( BDz = Tz \) which implies \( BDz = z \). Now we prove that \( Mz = z \). from (2), putting \( x = Mz \) and \( y = z \), we get

\[
d(Mz, z) \leq \alpha \left( \frac{d(Mz, Mz) + d(z, z)d(Mz, Mz)}{1 + d(Mz, z)} \right) + \beta \max\{d(Mz, BDz), d(Mz, Mz), d(BDz, S(Mz))\} + \gamma \{d(BDz, Tz) + d(Tz, AMz) + d(Sz, BDz)\} + \eta \left( \frac{d(Tz, Mz) + d(Sz, Mz)}{d(Mz, AMz)} \right)
\]

Therefore \( Mz = z \). since \( AMz = z \) which implies \( Az = z \). Now we prove that \( Dz = z \). from (2), putting \( x = z \) and \( y = Dz \), we get

\[
d(Dz, z) \leq \alpha \left( \frac{d(z, z) + d(Dz, Dz)d(z, Dz)}{1 + d(z, Dz)} \right) + \beta \max\{d(z, Dz), d(z, z), d(Dz, Dz)\} + \gamma \{d(Dz, Dz) + d(Dz, z) + d(z, Dz)\} + \eta \left( \frac{d(Dz, Dz) + d(Dz, z) + d(z, Dz)}{1 + d(z, Dz)} \right)
\]

Therefore \( Dz = z \). since \( BDz = z \) which implies \( Bz = z \).

Therefore by combining all the above result, we conclude that \( z \) is a common fixed point of \( A, B, D, M, S \) and \( T \).

The proof is similar when \( AM \) is continuous. Similarly, the result follows when (4) holds.

**Uniqueness:** Let \( u \) be another common fixed point of \( A, B, D, M, S \) and \( T \). Then, we have

\[
d(Sz, Tu) \leq \alpha \left( \frac{d(Az, Sz) + d(Bu, Tu)d(Az, Sz)}{1 + d(Sz, Tu)} \right) + \beta \max\{d(Az, Bu), d(Az, Sz), d(Bu, Sz)\} + \gamma \{d(Bu, Tu) + d(Tu, Amz) + d(Sz, Bu)\} + \eta \left( \frac{d(Tu, Amz) + d(Sz, Bu) + d(Bu, Amz)}{d(Tu, Amz) + d(Sz, Bu) + d(Bu, Amz)} \right)
\]

Therefore \( Dz = z \) is a unique common fixed point of \( A, B, D, M, S \) and \( T \).

**Corollary:** Let \( D, M, S \) and \( T \) be weakly reciprocally continuous self mapping of a complete complex valued metric space \( (X, d) \) satisfying conditions

1. \( S(X) \subset D(X) \) and \( T(X) \subset M(X) \)
2. For each \( x, y \in X \), where \( \alpha, \beta, \gamma \) and \( \eta \) are non negative real number with \( \alpha + \beta + 2\gamma + \eta \leq 1 \), such that
\[ d(Sx,Ty) \leq \frac{\alpha [d(Mx, Sx) + d(Dy, Ty) d(Mx, Sx)]}{1 + d(Sx, Ty)} + \beta \max\{d(Mx, Dy), d(Mx, Sx), d(Dy, Sx)\} \]
\[ + \gamma \left( d(Dy, Ty) + d(Ty, Mx) + d(Sx, Dy) \right) + \eta \left[ \frac{d(Ty, Dy) d(Sx, Mx)}{d(Ty, Mx) + d(Sx, Dy) + d(Dy, Mx)} \right] \]

3. \((M, S)\) are compatible, and \(M\) or \(S\) is continuous and \((D, T)\) are weakly compatible.
4. \((D, T)\) are compatible, and \(D\) or \(T\) is continuous and \((M, S)\) are weakly compatible.
Then \(D, M, S\) and \(T\) have a unique common fixed point.

**IV. Conclusion**

In this paper we proved fixed point theorem and common fixed point theorem in complex valued metric space through concept of compatibility and continuity.

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**References**