A NOTE ON VOLUME OF PARALLELIPIPED
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Abstract: Determinant gives an \(n\)-dimensional volume spanned by \(n\)-independent vectors in \(\mathbb{R}^n\). In this article we work out a method of finding the \(k\)-dimensional volume spanned by \(k\)-independent vectors in \(\mathbb{R}^n\) such that \(k \leq n\).

Keywords: Determinant, Parallelepiped, \(k\)-dimensional Volume

I. INTRODUCTION
Volume is scalar quantity associated with geometric objects. A parallelepiped is one of the simplest object in \(\mathbb{R}^n\). More complex objects may be approximated by these parallelepipeds.

Def (Parallelepiped): A parallelepiped \([u_1, \ldots, u_n]\) spanned by \(k\)-independent vectors \(v_1, \ldots, v_k\) is the set of all points in \(\mathbb{R}^n\) satisfying following condition:

\[
[v_1, \ldots, v_k] = \left\{ v \in \mathbb{R}^n \mid v = \sum_{i=1}^{k} \alpha_i v_i \mid 0 \leq \alpha_i \leq 1 \right\}
\]

Now, we can define the signed volume of parallelepiped with following properties, i.e.,

\[
\text{vol}: \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}
\]

1. If we multiply one side of parallelepiped with non-zero scalar \(\alpha\) then it’s volume will be multiplied by same amount.

\[
\text{vol}[v_1, \ldots, \alpha v_i, \ldots, v_n] = \alpha \text{vol}[v_1, \ldots, v_i, \ldots, v_n]
\]

2. \(\text{vol}[v_1, \ldots, v_i + v_j, \ldots, v_n] = \text{vol}[v_1, \ldots, v_n]\)

3. \(\text{vol}[e_1, \ldots, e_n] = 1\)
All these properties derived from the geometric idea. To compare it with determinant we need extra property of skew-symmetry.

4. \( \text{vol}\{v_1,\ldots,v_i,\ldots,v_j,\ldots,v_n\} = \text{vol}\{v_1,\ldots,v_j,\ldots,v_i,\ldots,v_n\} \)

Now all these property 1-4 satisfies the definition of determinant given by Curtis (1984).

Hence, \( \text{vol}\{v_1,\ldots,v_n\} = \det(v_1,\ldots,v_n) \)

**Theorem:** Let \( W \) be some \( k \)-dimensional subspace of \( \mathbb{R}^n \), then there exist some orthogonal matrix \( \mathbf{B} \) of \( \mathbb{R}^{k \times n} \), such that \( \text{vol}\{\mathbf{v}_1,\ldots,\mathbf{v}_n\} \). Extend this basis to form a basis of \( \mathbb{R}^n \).\n
**Proof:** Choose any orthonormal basis of \( W \) such that \( b_1,\ldots,b_k \). Extend this basis to form a basis of \( \mathbb{R}^n \), \( (b_1,\ldots,b_n) \). Now, Define a linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^n \), such that \( T(b_i) = e_i \).

This transformation is basically change of basis. To compute the matrix associated with respect to \( T \). Let’s compute \( T^{-1} \)

\[
T^{-1} : \mathbb{R}^n \to \mathbb{R}^n \\
T^{-1}(e_i) \to b_i \\
\]

Hence,

\[
\mathbf{B} = m(T^{-1}) = \begin{bmatrix} b_1 & b_2 & \ldots & b_n \end{bmatrix}
\]

\( \mathbf{B} \) is orthogonal matrix implies \( \mathbf{B}^{-1} \) is orthogonal and the required transformation.

**II. MAIN RESULT**

Volume of \( k \)-dimensional parallelepiped formed by \( k \)-independent vectors. \( x_1,\ldots,x_k \) will be given by the following formula \( V\{x_1,\ldots,x_k\} = \sqrt{|\det(x^T x)|} \) where \( x \) is \( n \times k \) dimensional matrix formed by columns of \( x \).

**Proof:** Given vectors \( x_1,\ldots,x_k, \exists A \in O(n) \). Such that \( Ax_i = \begin{bmatrix} y_i \\ 0 \end{bmatrix} \) such that \( y_i \in \mathbb{R}^k \)

Now, \( \text{vol}\{x_1,\ldots,x_k\} = \det\begin{vmatrix} y_1 & \ldots & y_k \end{vmatrix} = \det Y \)

Now, \( (Ax)^T (Ax) = Y^T Y \).

Taking, \( (x^T A^T Ax) = (Ax)^T (Ax) \)

\( \Rightarrow x^T x = Y^T Y \)

\( \Rightarrow \det(x^T x) = \det(Y)^T \det(Y) \)

\( \Rightarrow \det(x^T x) = \det(Y) \cdot \det(Y) \)

\( = [\det(Y)]^2 \)

\( \Rightarrow \det(Y) = \sqrt{\det(x^T x)} \)

Hence, \( \text{vol}\{x_1,\ldots,x_k\} = \sqrt{\det(x^T x)} \)

**III. CONCLUSION**

The volume of \( k \)-dimensional parallelepiped in terms of determinant is obtained. Determinants are relatively simple to calculate using Gaussian Elimination. Thus the volume of parallelepiped can be determined easily.

**REFERENCES**

