Almost $\eta$-Duals of Some Difference Sequence Spaces
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Abstract: Ansari and Shukla [3] have generalized the notion of $\alpha$-duals and developed the concept of almost $\alpha$-duals by using the concept of absolutely almost convergence. P Chandra and B.C. Tripathi [10] have introduced the concept of $\eta$-duals. Kizmaz [6] introduced the difference sequence spaces which was later on generalized by Sarigöl [11] and Ahmad & Mursaleen [1] and others Ansari and Gupta [2] have introduced the concept of almost $\eta$-duals. In this paper we have determined the almost $\eta$-duals of some difference sequence spaces.

Key Words: $\alpha$-duals, Almost $\alpha$-duals, $\eta$-duals, almost $\eta$-duals, difference sequence spaces

I. Introduction
The concept of almost convergence is introduced by Lorenz [8] and the concept of absolutely almost convergence is introduced by Das, Kuttener and Nanda [4]. Köthe - Toeplitz [7] defined $\alpha$ and $\beta$-duals in scalar case which was later generalized in operator version [9]. Ansari and Gupta [2] introduced the concept of almost $\eta$-duals using the concept of almost convergence is introduced by Das, Kuttener and Nanda [4]. Köthe - Toeplitz [7] defined $\alpha$ and $\beta$-duals in scalar case which was later generalized in operator version [9]. Ansari and Gupta [2] introduced the concept of almost $\eta$-duals using the concept of almost convergence. P. Chandra and B.C. Tripathi [10] have introduced the concept of almost $\eta$-duals developed by P. Chandra and B.C. Tripathi [10]. The difference sequence spaces is introduced by Kizmaz [6] which was later generalized by Sarigöl [11] and Ahmad & Mursaleen [1] and others. We have determined the almost $\eta$-duals of some difference sequence spaces in this paper.

Some Definitions and Relations:
The sequence spaces defined by Kizmaz [6] are
\[ l_{\infty} (\Delta) = \{ x = (x_k) : \Delta x \in l_{\infty} \}, \quad c (\Delta) = \{ x = (x_k) : \Delta x \in c \}, \]
\[ c_0 (\Delta) = \{ x = (x_k) : \Delta x \in c_0 \}, \text{ where } \Delta x = (\Delta x_k) = (x_k - x_{k+1}) \]

Above spaces are Banach spaces with the norm \( ||x|| = |x_1| + ||\Delta x||_\infty \), where \( ||\Delta x||_\infty = \sup_{k\geq 2} |x_k - x_{k+1}| \). For convenience, we denote these spaces by \( \Delta l_{\infty}, \Delta c \) and \( \Delta c_0 \) and call these constituent sequences by \( \Delta \)-bounded, \( \Delta \)-convergent, \( \Delta \)-null sequences, respectively.

Let \( E \) be any of the spaces \( l_{\infty}, c \) and \( c_0 \) then it is easy to see that \( E \subsetneq \Delta E \). Further, the containment is strict. For example. Let \( x_k = k, k = 1, 2, 3, \ldots \). Then the sequence \( (x_k) \notin c \) but \( (x_k) \in \Delta c \). Let \( p = (p_k) \) denote a sequence of strictly positive numbers (not necessarily bounded). Ahmad and Mursaleen [1] defined the sequence spaces
\[ \Delta l_{\infty} (p) = \{ x = (x_k) : \Delta x \in l_{\infty} (p) \} \]
\[ \Delta c (p) = \{ x = (x_k) : \Delta x \in c (p) \} \]
\[ \Delta c_0 (p) = \{ x = (x_k) : \Delta x \in c_0 (p) \} \]

when \( (p_k) \) is constant with all terms equal to \( p > 0 \), we have \( \Delta l_{\infty} (p) = l_{\infty} (\Delta), \Delta c (p) = c (\Delta) \) and \( \Delta c_0 (p) = c_0 (\Delta) \).

We make use in the proof of theorem (1) and (2) of the following Lemma whose proof can be found in Kizmaz [6].

Lemma 1. \( \sup_{k \geq 1} |x_k - x_{k+1}| < \infty \) iff

(i) \( \sup_{k \geq 1} k^{-1} |x_k| < \infty \) and
(ii) \( \sup_{k \geq 1} |x_k - k - 1| < \infty \)

Theorem 1. The almost $\eta$-dual of \( l_{\infty} (\Delta) \) is \( \{ l_{\infty} (\Delta) \}^\eta = \hat{D}_\eta \), where
\[ \hat{D}_\eta = \{ a = (a_k) : \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^s} \sum_{i=1}^{k} i (n + i) a_{n+i} \} < \infty \text{ uniformly for every } n \} \]

(1)
Proof: Let \((x_k) \in l_\infty(\Delta)\) and \(a = (a_k) \in \hat{D}_r\). Then

\[
\sum_{k=1}^{\infty} |\phi_{k,n}(x)| = \sum_{k=1}^{\infty} \left\| \sum_{i=1}^{k} a_{n+i} x_{n+i} \right\|^{\prime} = \sum_{k=1}^{\infty} \left\| \sum_{i=1}^{k} j(n+i) a_{n+i} x_{n+i} \right\|^{\prime}
\]

\[
\leq \sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=1}^{k} \left\| x_{n+i} \right\|^{\prime} \sup_{1 \leq i \leq k} \left\| j(n+i) a_{n+i} \right\|^{\prime}
\]

\[
\leq \sup_{1 \leq i \leq k} \left\| x_{n+i} \right\|^{\prime} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=1}^{k} j(n+i) a_{n+i}
\]

\[
< \infty, \text{ for every } n \{ \text{using lemma (1) and equation (1)} \}
\]

Thus \(\sum_{k=1}^{\infty} |\phi_{k,n}(x)|^{\prime} < \infty, \text{ for all } n\). Hence, \(a = (a_k) \in [l_\infty(\Delta)]^{\hat{n}}\). Therefore \(\hat{D}_r \subset [l_\infty(\Delta)]^{\hat{n}}\).

Conversely, let us suppose that \(a = (a_k) \in [l_\infty(\Delta)]^{\hat{n}}\) but \(a_k \not\in \hat{D}_r\)

\[
i.e. \sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=1}^{k} j(n+i) a_{n+i}^{\prime} = \infty, \text{ for some } n
\]

Choose \(x = x_k \) with \(x_k = k \text{ sgn } a_k\). Then \(x \in l_\infty(\Delta)\)

But,

\[
\sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=1}^{k} j(n+i) x_{n+i}^{\prime} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=1}^{k} j(n+i) a_{n+i}^{\prime} = \infty
\]

which is contradiction to the fact that \((a_k) \in [l_\infty(\Delta)]^{\hat{n}}\).

Therefore \([l_\infty(\Delta)]^{\hat{n}} \subset \hat{D}_r\). Thus \([l_\infty(\Delta)]^{\hat{n}} = \hat{D}_r\). This completes the proof.

Theorem 2. For every strictly positive sequence \(p = (p_k)\), we have \([A_{\infty}(p)]^{\hat{n}} = \hat{D}_r(p)\), where

\[
\hat{D}_r(p) = \bigcap_{n>1} \{a = (a_k) : \sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=1}^{k} j(n+i) a_{n+i} N^{b_i} \left| a_{n+i} \right| \left\| a_{n+i} \right\|^{\prime} < \infty, \text{ uniformly in } n\} \quad \cdots(2)
\]

Proof:

Let \(a = (a_k) \in \hat{D}_r(p)\) and \(x = (x_k) \in \Delta l_\infty(p)\). We choose an integer \(N > \max \{1, \sup_{k=1}^{\infty} \frac{|x_k|}{k} \}\). Then

\[
\sum_{k=1}^{\infty} |\phi_{k,n}(x)|^{\prime} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=1}^{k} \left\| j(n+i) a_{n+i} x_{n+i} \right\|^{\prime} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=1}^{k} j(n+i) a_{n+i} x_{n+i}
\]

\[
< N^{b_i} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=1}^{k} j(n+i) a_{n+i}
\]

\[
< \infty
\]

Therefore \(a = (a_k) \in [A_{\infty}(p)]^{\hat{n}}\) for all \(n\) (using equation (2))
Hence $\hat{D}_r (p) \subset [\mathcal{L}_\infty (p)]^\hat{\Pi}$.

Converse, now suppose that $a = (a_k) \in [\mathcal{L}_\infty (p)]^\hat{\Pi}$

but $a = (a_k) \notin \hat{D}_r (p)$

$\Rightarrow$ There is an integer $N > 1$ such that

$$\sum_{k=1}^{\infty} \frac{1}{k'} \left| \sum_{i=1}^{k} i(n+i) a_{n+i} N ^{-\frac{1}{p}} \right|^{l} = \infty$$

choose $x = (x_k)$ such that $x_k = \frac{1}{a_k} \cdot \hat{D}_r (p)$, then $x \in \Delta \mathcal{L}_\infty (p)$ but

$$\sum_{k=1}^{\infty} \frac{1}{k'} \left| \sum_{i=1}^{k} i a_{n+i} x_{n+i} \right|^{l} = \sum_{k=1}^{\infty} \frac{1}{k'} \left| \sum_{i=1}^{k} i a_{n+i} (n+i) N ^{-\frac{1}{p}} \right|^{l} \cdot \text{sgn}(a_{n+i})$$

$$= \sum_{k=1}^{\infty} \frac{1}{k'} \left| \sum_{i=1}^{k} i(n+i) \left| a_{n+i} \right| \left| N ^{-\frac{1}{p}} \right|^{l} \right| = \infty$$

which is contradiction that $a = (a_k) \in [\mathcal{L}_\infty (p)]^\hat{\Pi}$. Hence $[\mathcal{L}_\infty (p)]^\hat{\Pi} \subset \hat{D}_r (p)$

Thus, $[\mathcal{L}_\infty (p)]^\hat{\Pi} = \hat{D}_r (p)$ which completes the proof.

**Generalized Almost $\eta$-dual of $l_\infty (A^m)$**: Et and colak [5] defined the sequence spaces

$$l_\infty (A^m) = \{ x = (x_k) : A^m x \in l_\infty \}$$

$$c (A^m) = \{ x = (x_k) : A^m x \in c \}$$

$$c_0 (A^m) = \{ x = (x_k) : A^m x \in c_0 \}$$

where $m \in N$, $A^m x = x_k$, $A^m x_k = \Delta x_k (x_k - x_{k+1})$

$$\Delta^m (x) = (A^m x_k) = (A^m-1 x_k - A^m-1 x_{k+1})$$

These spaces are Banach spaces with the norm

$$\| x \|_{A^m} = \sum_{i=1}^{m} \| x_i \| + \| A^m x \|_{\infty}$$

where, $\| A^m x \|_{\infty} = \sup_{k \geq m} | A^m x_k - A^m x_{k+1} |$

In this section, we have determined the almost $\eta$-dual of $l_\infty (A^m)$.

We make use in the proof of theorem [3] of the following lemmas whose proof can be found in Et and colak [5].

**Lemma 2**: $\sup_{k \geq m} | A^m x_k - A^m x_{k+1} | < \infty$ iff

(i) $\sup_{k \geq 1} k^{-1} | A^m x_k | < \infty$ and (ii) $\sup_{k \geq 1} k^{-1} | A^m x_k - k(k+1)^{-1} A^m x_{k+1} | < \infty$.

**Lemma 3**: $x \in l_\infty (A^m)$ implies that $\sup_{k} k^{-m} \left| x_k \right| < \infty$.

**Theorem 3**: The almost $\eta$-dual of $l_\infty (A^m)$ is $\left[ l_\infty (A^m) \right]^\eta = \hat{U}_r$ where

$$\hat{U}_r = \{ a = (a_k) : \sum_{k=1}^{\infty} \frac{1}{k'} \left| \sum_{i=1}^{k} i \cdot (n+i)^m a_{n+i} \right|^{l} < \infty \text{ uniformly for all } n \}. \quad (3)$$

**Proof**: Let $a = (a_k) \in \hat{U}_r$ and $x = (x_k) \in l_\infty (A^m)$. Then

$$\sum_{k=1}^{\infty} \left| \phi_{k,n} \right|^{l} = \sum_{k=1}^{\infty} \frac{1}{k'} \left| \sum_{i=1}^{k} i a_{n+i} x_{n+i} \right|^{l} = \sum_{k=1}^{\infty} \frac{1}{k'} \left| \sum_{i=1}^{k} i(n+i)^m a_{n+i} \frac{x_{n+i}}{(n+i)^m} \right|^{l}$$
\[
\leq \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \sup_{i \leq k} \left| \frac{x_{n+i}}{(n+i)^m} \sum_{j=1}^{k} (n+j)^m a_{n+j} \right|^r
\]
\[
\leq \sup_{i \geq 1} \left| \frac{x_i}{i^m} \right| \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \sum_{j=1}^{k} i(n+i)^m a_{n+i} \right|^r < \infty \quad \text{(By Lemma (2) and equation (3)).}
\]

Hence \(a = (a_k) \in \left[ \ell^r_{\infty}(\Delta^m) \right]^\mathbb{N}\). Therefore \(\hat{U}_r \subseteq \left[ \ell^r_{\infty}(\Delta^m) \right]^\mathbb{N}\).

Converse, let \(a = (a_k) \in \left[ \ell^r_{\infty}(\Delta^m) \right]^\mathbb{N}\). Then \(\sum_{k=1}^{\infty} |\phi_{k,n}|^r < \infty\) for each \(x = (x_k) \in \ell^r_{\infty}(\Delta^m)\).

For the sequence \(x = (x_k)\) defined by
\[
x_k = k^m, \quad k > m
\]
\[
x_k = 0, \quad k \leq m
\]
Then, \(x = (x_k) \in \ell^r_{\infty}(\Delta^m)\)
and hence \(\sum_{k=1}^{\infty} |\phi_{k,n}|^r = \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^{k} i a_{n+i} x_{n+i} \right|^r < \infty,\)
\[
\sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^{k} i(n+i)^m a_{n+i} \right|^r
\]
\[
= \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^{k} i(n+i)^m a_{n+i} \right|^r + \sum_{k=m+1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^{k} i(n+i)^m a_{n+i} \right|^r
\]
\[
\leq \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^{k} i(n+i)^m a_{n+i} \right|^r + \sum_{k=1}^{\infty} |\phi_{k,n}|^r < \infty.
\]

Hence \(a = (a_k) \in \hat{U}_r\). Therefore, \(\left[ \ell^r_{\infty}(\Delta^m) \right]^\mathbb{N} \subseteq \hat{U}_r\).

Thus, \(\left[ \ell^r_{\infty}(\Delta^m) \right]^\mathbb{N} = \hat{U}_r\).

**Generalized almost \(\eta\)-dual of \(l^r_{\infty}(\Delta^m)\):** Let \(v = (v_k)\) be a sequence of non-zero complex numbers such that
\[
\frac{|v_k|}{|v_{k+1}|} + 1 = 0 (1/k) \text{ for each } k \quad \text{and} \quad k^{-1} |v_k| \sum_{i=1}^{k} |v_i|^{-1} = 0 (1)
\]
and \(\left( |k|, |v_k|^{-1} \right)\) is a sequence of positive numbers increasing monotonically to infinity.

Srivastava and Gnanaseelan [12] defined \(F(\Delta^m) = \{x = (x_k) : (v_k (x_k - x_{k+1})) \in F\}\), where \(F = l^r_{\infty}, c \text{ or } c_0\).

Obviously, these spaces are linear spaces. Further, it is easy to verify that
(i) \(F(\Delta^m) \subseteq F(\Delta)\) if \(|\Delta_k| \geq 1\) and \(|v_k|\) increases to \(\infty\).
(ii) \(F(\Delta) \subseteq F(\Delta^m)\) if \(|\Delta| < 1\) and \(|v_k|\) decreases to \(0\). These spaces are Banach spaces with the norm
\[
\|x\|_{\Delta^m} = \|v\|_{1 \times 1} + \sup_{k} |v_k (x_k - x_{k+1})|
\]

In this section, we have determined the almost \(\eta\)-dual of \(l^r_{\infty}(\Delta^m)\).

We make use in the proof of the theorem (4) of the following Lemma whose proof can be found in Srivastav and Gnanaseelan [4].

**Lemma : 4.** \(\sup_{k \geq 1} |x_k| < \infty\) iff
(i) \sup_{k \geq 1} |x_k| < \infty and (ii) \sup_{k \geq 1} |v_k (x_k - k (k + 1)^{-1} x_{k+1})| < \infty.

**Theorem : 4.** The almost \(\eta\)-dual of \(l^r_{\infty}(\Delta^m)\) is \(\left[ l^r_{\infty}(\Delta^m) \right]^\mathbb{N} = \hat{D}_{r}(r)\), where
\[ \hat{D}_v(r) \quad (a = (a_k)) : \sum_{k=1}^{\infty} \frac{1}{k^r} \left( \sum_{i=1}^{k} i(n+i)\nu_{n+i}^{-1}a_{n+i} \right)^r < \infty \text{ uniformly for every } n \text{ and } 0 < r \leq 1 \] ...

**Proof:** Let \( x = (x_k) \in L_\infty(\Delta_v) \) and \( a = (a_k) \in \hat{D}_v(r) \). Then

\[
\sum_{k=1}^{\infty} \frac{1}{k^r} \left( \sum_{i=1}^{k} i(n+i)\nu_{n+i}^{-1}a_{n+i} \right)^r = \sum_{k=1}^{\infty} \frac{1}{k^r} \left( \sum_{i=1}^{k} i(n+i)\nu_{n+i}^{-1}a_{n+i} \right)^r 
\]

\[
\leq \sup_{1 \leq i \leq n} \left| \frac{1}{k^r} \left( \sum_{i=1}^{k} i(n+i)\nu_{n+i}^{-1}a_{n+i} \right)^r \right| \]

\[
< \infty \quad \text{[using Lemma (4) and equation (4)]}
\]

Thus \( \sum_{k=1}^{\infty} |\phi_{k,n}|^r < \infty \) for all \( n \). Hence \( a = (a_k) \in \left[ L_\infty(\Delta_v) \right]^n \).

Therefore, \( \hat{D}_v(r) \subset \left[ L_\infty(\Delta_v) \right]^n \).

Conversely, let us suppose that \( a = (a_k) \in \left[ L_\infty(\Delta_v) \right]^n \) but \( a_k \notin \hat{D}_v(r) \) i.e.

\[
\sum_{k=1}^{\infty} \frac{1}{k^r} \left( \sum_{i=1}^{k} i(n+i)\nu_{n+i}^{-1}a_{n+i} \right)^r = \infty, \text{ for some } n. \text{ Choose } x = (x_k) \text{ with } x_k = k \nu_k^{-1} |\text{sgn} a_k|.
\]

Then, \( (x_k) \in L_\infty(\Delta_v) \), because

\[
|v_k(x_k - k (k+1)^{-1} x_{k+1})| = |v_k k |\text{sgn} a_k - k| |\text{sgn} a_{k+1}|
\]

\[
= |k \text{sgn} a_k - k \nu_k^{-1} v_k| |\text{sgn} a_{k+1}|
\]

\[
= k \nu_k^{-1} |v_k| |\text{sgn} a_k|
\]

\[
= k \left( \frac{1}{k} \right)
\]

\[
= 0 \quad (1)
\]

Thus \( \sup_{k \geq 1} |v_k - k (k+1)^{-1} x_{k+1}| < \infty \) and

\[
\sup_{k \geq 1} |v_k x_k| = \sup_{k \geq 1} k^{-1} |v_k k \nu_k^{-1} \text{sgn} a_k| = \sup_{k \geq 1} |\text{sgn} a_k| = 1 < \infty.
\]

Then by Lemma (4), we have \( \sup_{k \geq 1} |v_k (x_k - x_{k+1})| < \infty \).

But

\[
\sum_{k=1}^{\infty} \frac{1}{k^r} \left( \sum_{i=1}^{k} i(n+i)\nu_{n+i}^{-1}a_{n+i} \right)^r = \sum_{k=1}^{\infty} \frac{1}{k^r} \left( \sum_{i=1}^{k} i(n+i)\nu_{n+i}^{-1} |a_{n+i}| \right)^r = \infty
\]
which is contradiction that \( a = \{a_k\} \in \left[l_\infty(\Delta_v)\right]^\Delta \)

Therefore, \( \left[l_\infty(\Delta_v)\right]^\Delta \subset \hat{D}_s(r) \)

Thus \( \left[l_\infty(\Delta_v)\right]^\Delta \subset \hat{D}_s(r) \)

This completes the proof.

**Semi Difference sequence space and their \( \eta \)-dual :** Let \( w \) be the space of all complex valued sequences \( x = (x_k) \). Kizmaz [6] introduced the difference sequence space.

\[ l_\infty(\Delta) = \{x = (x_k) \in w : (\Delta x_k)^{\infty}_{k=1} \in l_\infty\} \]

He further proved that a sequence \( (x_k) \in l_\infty(\Delta) \) iff

\[ (i) \quad \sup_{k \geq 1} \left| \frac{x_k}{k} \right| < \infty \quad \text{and} \quad (ii) \quad \sup_{k \geq 1} \left| x_k - k(k+1)^{-1}x_{k+1} \right| < \infty \]

By using above two conditions.

Ansari ans Shukla [3] introduced a new sequence space and call it to be a semi difference sequence space. Thus a semi difference sequence \( (x_k) \) is defined as a sequence \( (x_k) \) such that \( \sup_{k \geq 1} \left| \frac{x_k}{k} \right| < \infty \) or a semi difference sequence space

\[ l_\infty(S \Delta) = \{x = (x_k) \in w : \sup_{k \geq 1} \left| \frac{x_k}{k} \right| < \infty\} \quad \ldots(5) \]

Ansari and Shukla [3] have proved that \( l_\infty(S \Delta) \) is a Banach space with norm defined by

\[ \|x\| = \sup_{k \geq 1} \left| \frac{x_k}{k} \right| \]

In this section, we have determined the almost \( \eta \)-dual of \( l_\infty(\leq \Delta) \).

**Theorem : 5.** The \( \eta \)-dual of \( l_\infty(\leq \Delta) \) is \( l_\infty^\eta(s\Delta) = D_\delta(r) \) where

\[ D_\delta(r) = \{ a = \{a_k\} \in w : \sum_{k=1}^\infty k^r \left| a_k \right|^r < \infty \quad \text{and} \quad 0 < r \leq 1 \}. \quad \ldots(6) \]

**Proof :** Let \( a = (a_k) \in D_\delta(r) \) and \( x = (x_k) \in l_\infty(s\Delta) \) Then,

\[ \sum_{k=1}^\infty \left| a_k x_k \right|^r = \sum_{k=1}^\infty \left| a_k \right|^r \left| x_k \right|^r < \left( \sum_{k=1}^\infty \left| a_k \right|^r \right) \left( \sum_{k=1}^\infty \left| x_k \right|^r \right) < \infty \quad \text{using equations (5) and (6)} \]

Thus \( a = \{a_k\} \in l_\infty^\eta(s\Delta) \). Hence \( D_\delta(r) \subset l_\infty^\eta(s\Delta) \).

Conversely, If \( a = \{a_k\} \in l_\infty^\eta(s\Delta) \)

\[ \Rightarrow \sum_{k=1}^\infty \left| a_k x_k \right|^r < \infty \text{ for each } x = (x_k) \in l_\infty(s\Delta) \]

Define \( x = (x_k) \), where \( x_k = k \), for each \( k \). Then \( x = (x_k) \in l_\infty(s\Delta) \)

\[ \sum_{k=1}^\infty \left| a_k x_k \right|^r = \sum_{k=1}^\infty \left| a_k \right|^r \left| x_k \right|^r = \sum_{k=1}^\infty k^r \left| a_k \right|^r < \infty \]

Thus \( a = \{a_k\} \in D_\delta(r) \). Hence \( l_\infty^\eta(s\Delta) \subset D_\delta(r) \).

Therefore, \( l_\infty^\eta(s\Delta) = D_\delta(r) \), which completes the proof.

**Theorem : 6.** The almost \( \eta \)-dual of \( l_\infty(\leq \Delta) \) is \( l_\infty^\eta(s\Delta) = \hat{D}_\delta(r) \), where

\[ \hat{D}_\delta(r) = \{ a = \{a_k\} : \sum_{k=1}^\infty \frac{1}{k^r} \sum_{i=1}^k i a_{n+i}(n+i)^r \} < \infty \text{ uniformly in } n, \text{ for and } 0 < r \leq 1 \}. \quad \ldots(7) \]

**Proof :** Let \( x = (x_k) \in l_\infty(s\Delta) \) and \( a = \{a_k\} \in \hat{D}_\delta(r) \). Then,
\[
\sum_{k=1}^{\infty} \left| \phi_{k,n} \right|^r = \sum_{k=1}^{\infty} \sum_{i=1}^{k} \frac{1}{k^r (k+1)^r} \left| \sum_{j=1}^{k} i a_{n+i} x_{n+i} \right|^r
= \sum_{k=1}^{\infty} \sum_{i=1}^{k} \frac{1}{k^r (k+1)^r} \left| \sum_{j=1}^{k} i a_{n+i} (n+i) \frac{x_{n+i}}{n+i} \right|^r
\leq \sum_{k=1}^{\infty} \sum_{i=1}^{k} \frac{1}{k^r (k+1)^r} \left| \sum_{j=1}^{k} i a_{n+i} (n+i) \right|^r
\leq \sum_{i=1}^{\infty} \frac{1}{i^r} \sum_{k=1}^{\infty} \sum_{i=1}^{k} \frac{1}{k^r (k+1)^r} \left| \sum_{j=1}^{k} i a_{n+i} (n+i) \right|^r
< \infty, \text{ for each } n. \]
Thus \( \sum_{k=1}^{\infty} \left| \phi_{k,n} \right|^r < \infty, \) for each \( n. \) Therefore, \( \hat{D}_x(r) \subset \left[ l_{\infty}(s\Delta) \right]^h \)

converse, Let us suppose that \( a = (a_k) \in \left[ l_{\infty}(s\Delta) \right]^h \)

But \( a = (a_k) \not\in \hat{D}_x(r) \) i.e. \( \sum_{i=1}^{\infty} \left| \sum_{j=1}^{k} (n+i) a_{n+i} \right|^r = \infty, \) for some \( n. \)

Choose \( x = (x_k) \) with \( x_k = k, \) for each \( k. \) Then \( x = (x_k) \in l_{\infty}(s\Delta) \)

but \( \sum_{i=1}^{\infty} \sum_{j=1}^{k} \frac{1}{i^r} \left| \sum_{j=1}^{k} i a_{n+i} x_{n+i} \right|^r
= \sum_{k=1}^{\infty} \sum_{i=1}^{k} \frac{1}{i^r} \left| \sum_{j=1}^{k} i (n+i) a_{n+i} \right|^r
= \infty \)

which is contradiction to the fact that \( a = (a_k) \in \left[ l_{\infty}(s\Delta) \right]^h \)

Hence, \( \left[ l_{\infty}(s\Delta) \right]^h \subset \hat{D}_x(r). \) Thus \( \left[ l_{\infty}(s\Delta) \right]^h = \hat{D}_x(r). \)
which completes the proof.

References