A Non-differential Approach for Solving a Class of Linear and Non-linear Bi-Level Programming Problems Based on Analytic Hierarchy Process

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Abstract: Bi-level programming is a tool for modeling decentralized decisions. In this paper we consider the solution of a bi-level linear and non-linear programming problem by weighting method. A non-dominated solution set is obtained by this method. In this process decision makers (DMs) provide their preference bounds to the decision variables that is the upper and lower bounds to the decision variables they control. We convert the hierarchical system into scalar optimization problem (SOP) by finding proper weights using the analytic hierarchy process (AHP) so that objective functions of both levels can be combined into one objective function. Here the relative weights represent the relative importance of the objective functions. Finally, the feasibility and effectiveness of the proposed approach is demonstrated by the numerical example.

Keywords: Bi-level programming problems, Optimization problem, Analytic hierarchy process, Optimal solution, Satisfactory Solution.

I. Introduction

A bi-level programming problem (BLPP) consists of two levels, namely, the first level and the second level. The first level decision maker (DM) is called the centre. The second level DM called follower, executes its policies after the decision of higher level DM called leader (centre) and then the leader optimizes its objective independently but may be affected by the reaction of the follower i.e. BLPP is a sequence of two optimization problems in which the constraints region of one is determined by the solution of second.

There are many methods to solve BLPPs. The formulation and different version of BLPP are given by Bard [10, 11], Candler [24], Bard and Falk [9] and Bialas and Karwan [23]. Bialas and Karwan [23] are the pioneers for linear BLPP who presented vertex enumeration method, called Kth-best solution. These were solved by simplex method. To solve the non-linear problem that arises due to the K-T conditions, Bialas and Karwan [23] proposed a parametric complementary pivot (PCP) algorithm which interactively solves a slight perturbation of the system. Bard and Falk [9] proposed the grid search algorithm. Based on Bard and Falk’s algorithm, Unlu [6] proposed a technique of bi-criteria programming. Mishra and Ghosh [19] presented fuzzy programming approach to solve bi-level linear fractional programming problems. Again, Mishra and Ghosh [17] proposed interactive fuzzy programming approach to solve bi-level quadratic fractional programming problems. Also, Mishra [18] presented weighting method for bi-level linear fractional programming problems. Linear and non-linear programming [1,17,18,19,20] is generally used for modelling real life problems. Fractional programs arise in various contexts such as, in investment problems, the firm wants to select a number of projects on which money is to be invested so that the ratio of the profits to the capital invested is maximum subject to the total capital available and other economic requirements which may be assumed to be linear or non-linear. Bi-level non-linear programming problems are studied by a few [17,20]. In this paper we deal with the linear and non-linear BLPPs with the essentially cooperative DMs and propose a solution procedure using analytic hierarchy process (AHP) for the problem.

In this paper we consider the solution of a linear and non-linear BLPP by Weighting method. In Weighting method decision makers (DMs) provide their preference bound to the decision variables that is the upper and lower bounds to the decision variables they control. We convert the hierarchical system into scalar optimization problem (SOP) by finding proper weights using AHP so that objective functions of both levels can be combined into one objective function. Here, the relative weights represent the relative importance of the objective functions. A non-dominated solution set is obtained by this method. Perhaps the most creative task in making a decision is to choose the factors that are important for that decision. In the AHP we arrange these factors, once selected, in a hierarchical structure descending from an overall goal to criteria, sub-criteria and alternatives in successive levels. This paper demonstrates the merit of this technique in deciding optimal solution.
of bi-level linear and non-linear decision-making problem taking into consideration the various constraints and complexities representing the real situation.

II. Linear and Non-linear Bi-level Programming Problems

A linear and non-linear BLPP consists of two levels, namely, the first level and the second level and each has linear and non-linear objective function. These divisions are independent. A multi-level programming problem (MLPP) can be defined as a p-person, nonzero sum game with perfect information in which each player moves sequentially from top to bottom. This problem is a nested hierarchical structure. When \( p = 2 \), we call the system a BLPP.

A bi-level linear and non-linear programming problem is formulated as:

\[
\begin{align*}
\text{maximize} & \quad z_1(x_1, x_2) \quad \text{where } x_1 \text{ solves} \\
\text{maximize} & \quad z_2(x_1, x_2) \\
\text{subject to} & \quad A x_1 + B x_2 \leq b \\
& \quad x_1 \geq 0, x_2 \geq 0.
\end{align*}
\]

Where objective functions \( z_i(x_1, x_2), \ i = 1,2 \) are represented by a linear and non-linear function

Also let DM1 denote the DM at the upper level and DM2 denote the DM at the lower level.

In the bi-level linear and non-linear programming problem, \( z_1(x_1, x_2) \) and \( z_2(x_1, x_2) \) respectively represent objective functions of DM1 and DM2 and \( x_1 \) and \( x_2 \) represent decision variables of DM1 and DM2 respectively.

III. Weighting Method

The basic idea of assigning weights to the various objective functions, combining these into a single objective function and parametrically varying the weights to generate the non-dominated set was first proposed by Zadeh in 1963. Mathematically, the weighting method can be stated as follows:

\[
\begin{align*}
\text{max/ min} & \quad w_1 z_1(\bar{x}) + w_2 z_2(\bar{x}) + \ldots + w_p z_p(\bar{x}) \\
\text{subject to} & \quad \bar{x} \in X \quad \text{where } X \text{ is the feasible region.}
\end{align*}
\]

Thus, a multiple objective problem has been transformed into a single objective optimization problem for which solution methods exist. The coefficient \( w_p \) operating on the \( p^{th} \) objective function, \( z_p(\bar{x}) \), is called the weight and can be interpreted as “the relative weight or worth” of that objective function when compared to the other objectives. These weights can be obtained by Analytic Hierarchy Process (AHP) [8,14,21]. If the weights of the various objectives are interpreted as the representing the relative preference of some DM, then the solution to (5) is equivalent to the best compromise solution, i.e., the optimal solution relative to a particular preference structure. Moreover, the optimal solution to (5) is a non-dominated solution provided all the weights are positive. Allowing negative weights would be equivalent to transforming the maximizing problem to a minimizing one, for which a different set of non-dominated solutions will exist. The trivial case where all the weights are zero will simply identify \( \bar{x} \in X \) as an optimal solution and will not distinguished between dominated and non-dominated solutions [3].

IV. Method for Generating Non-dominated Solutions

A non-dominated solution is one in which no one objective function can be improved without a simultaneous detriment to at least one of the other objectives of the vector maximum problem [VMP]. A given multiple objective mathematical problem which contains only maximization type objective functions is called the VMP. A feasible solution \( \bar{x}^* \in X \) (decision space) is a non-dominated solution to the VMP iff there does not exist any other feasible solution \( \bar{x} \in X \) such that \( z_p(\bar{x}^*) \leq z_p(\bar{x}), \ p = 1,2,\ldots, P \) and \( z_p(\bar{x}^*) < z_p(\bar{x}) \), for at least one \( P \).

The thrust of MODM models is to design the ‘best’ alternative by considering the various interactions within the design constraints that best satisfy the DM by way of obtaining some acceptable levels of quantifiable objective functions. The common characteristics of MODM methods are that, they possess:

* A set of quantifiable objective functions,
*A set of well defined constraints,
*A process of obtaining some tradeoff information, implicit or explicit, between the stated quantifiable objective functions and also between stated or un-stated non-quantifiable objective functions. The MODM problem is solved by adopting different techniques based on non-dominated solutions generation method. Different generating methods provide DMs tradeoff information in different forms. Li et al. [5] have characterized various generating methods and established the quantitative parametric connections between these generating methods.

Li also proves in his paper that applying the \( p^k \) power to the objective functions of a non-convex MODM problem can act as a convexification scheme for a non-inferior frontier. Then Li and Biswal [4] established the same results by taking the exponentials of the objective functions. This convexification further leads to the exponential generating method that guarantees the identification of the entire set of non-dominated solutions.

If we have a MODM problem with maximizing objective functions, then a non dominated solution is one for which there is no other solution giving equal or greater values of each and every objective function. But in even the smallest problem, the number of non-dominated solutions generated may be infinite. This is because all points on the line joining two non-dominated and extreme points are themselves non-dominated. The concept of non-dominated solution was introduced by Pareto, an economist in 1896. A preferred solution is a non-dominated solution which is chosen by the DM, through some additional criteria, as the final decision. As such it lies in the region of acceptance of all the criteria values for the problem. A preferred solution is also known as the ‘best’ solution.

The most common strategy for finding non-dominated solutions of a MODM problem is to convert it into a scalar optimization problem (SOP). There are a few methods available in the MODM literature:

1. The weighting method,
2. The \( k^{th} \) - objective Lagrangian method,
3. The \( k^{th} \) - objective \( \sigma \) - constraint method etc.

This class of methods does not require any assumption or information regarding the DMs utility function.

V. Weighting Method for Linear and Non-linear BLPP

In the weighting problem \( \bar{P}(w) \) in the absence explicit preference structure, the strategy is to generate all or representative subsets of non-dominated solutions from which a DM can select the suitable solution.

Let a linear and non-linear BLPP be represented as:

\[
\begin{align*}
\max_{x_1} & \quad z_1(\bar{x}) \\
\max_{x_2} & \quad z_2(\bar{x}) \\
\text{subject to} & \quad g_i(\bar{x}) \leq b_i, \quad i = 1,2,\ldots, m; \quad \bar{x}_1 \geq 0, \bar{x}_2 \geq 0,
\end{align*}
\]

Where, \( z_1(\bar{x}), z_2(\bar{x}), \) and \( g_i(\bar{x}) \), are linear and non-linear objective functions and linear or non linear constraints respectively. \( \bar{x}_1, \bar{x}_2 \) are decision vectors under the control of the upper and the lower level DM.

Let \( X = \) set of feasible solutions = \( \{ \bar{x} : \bar{x} \in \mathbb{R}^n, \quad g_i(\bar{x}) \leq b_i \} \),

\( \bar{x} = \) decision vector in n-dimensional Euclidean space = \( \bar{x}_1 \cup \bar{x}_2 \)

Any MODM solution procedure cannot be directly apply to BLPP, since in BLPP the DMs are on different hierarchical levels who control only a subset of the decision variables. DMs provide their preference bounds to the decision variables, i.e., the upper and lower bounds to the decision variables they control. Thus first convert the hierarchical system into a SOP by finding proper weights using the analytic hierarchy process (AHP) [21] so that objective functions on different levels can be combined into one objective function. Here the relative weights represent the relative importance of the objective functions. We use the AHP to establish the weights of the objective function in a BLPP. There are several other methods available in literature to find out weights from the pair wise comparison matrices, among the most mentionable are:

1. least squares method [16]
2. Logarithmic least squares method [7]

Solving the SOP involves finding \( \bar{x}^* \in X \) such that \( z(\bar{x}^*) \geq z(\bar{x}) \quad \forall \quad \bar{x} \in X \). The point \( \bar{x}^* \) is said to be global optimum. If strict inequality holds for the objective functions, then \( \bar{x}^* \) is the unique global optimum. If the
inequality holds for some neighborhood of \( \bar{x}^* \), then \( \bar{x}^* \) is a local or relative optimum while it is strict local optimum if strict inequality holds in a neighborhood of \( \bar{x}^* \).

In the weighting problem \( P(\bar{w}) \), in the absence of an explicit preference structure, the strategy is to generate all or representative subsets of non-dominated solutions from which a DM can select the suitable solution. The weighting problem for bi-level linear and non-linear programming problem is formulated as follows:

\[
P(\bar{w}) = \max_{\bar{x} \in X} \sum_{p=1}^{P} \bar{w}_p z_p(\bar{x})
\]

subject to

\[
g_i(x) \leq b_i, \quad i = 1,2,\ldots, m
\]

\[
L_i \leq \bar{x}_i \leq U_i
\]

\[
\bar{L}_2 \leq \bar{x}_2 \leq \bar{U}_2
\]

Where,

\[
\bar{w} \in W = \{ \bar{w} : \bar{w} \in R^p, \quad w_p \geq 0, \quad p = 1,2 \quad \text{and} \quad \sum_{p=1}^{P} w_p = 1 \}
\]

and \( z_p(x_1, x_2) \) is linear and non-linear objective functions.

\( \bar{U}_p \) and \( \bar{L}_p \) are the upper and lower bounds of decision vector provided by the respective DM. Finally the linear and non-linear programming problem (9) - (12), with a single objective function is solved. Here the weighting coefficients convey the importance attached to an objective function. Suppose that the relative importance of the both objective functions is known and is constant. Then the preferred solution is obtained by solving \( p(\bar{w}) \) where \( \bar{w}_p \)'s \( \geq 0 \) are the weighting coefficients. The \( \bar{w}_p \)'s are normalized since, \( \sum_{p=1}^{P} \bar{w}_p = 1 \).

This method can be used to generate non-dominated solutions by utilizing various values of \( \bar{w} \). In such a case the weighting coefficients \( \bar{w} \) do not reflect the relative importance of the objective functions in the proportional sense, but are only parameters varied to locate the non-dominated points [22].

VI. Numerical Examples

In this section we present numerical examples to demonstrate the solution procedures by proposed Weighting method to solve BLPP.

Example-1: Consider the following BLPP

\[
\begin{align*}
\max z_1 &= -x_1^2 + 6x_1x_2 + x_2 + x_1x_4x_5 \\
\max z_2 &= x_2^2 + x_3x_4x_5
\end{align*}
\]

subject to

\[
\begin{align*}
x_1 - x_2 + x_3 + x_4 - x_5 &\leq 40; \\
(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 - (x_4 - x_5)^2 &\geq 25; \\
(x_1 - 4)^2 + x_2^2 &\leq 16; \\
(x_3 - 3)^2 + (x_4 - 2)^2 &= 13; \\
x_1 &\geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0.
\end{align*}
\]

Solution: Let the bounds provided by the respective DMs be \( 5 \leq x_1 \leq 9.5, \quad 0 \leq x_2 \leq 3, \quad 2 \leq x_3 \leq 6.5, \quad 3.5 \leq x_4 \leq 7, \quad 10 \leq x_5 \leq 14 \). The pair wise comparison matrix \( \tilde{A} \) of order 2 and its normalized matrix \( \tilde{N}[25] \) for the hierarchical objective functions are given as:

\[
\tilde{A} = z_1 \begin{bmatrix} 1 & 5 \\ 1/5 & 1 \end{bmatrix} z_1
\]

\[
\tilde{N} = \begin{bmatrix} 1/1.2 & 5/6 \\ 0.2/1.2 & 1/6 \end{bmatrix} = \begin{bmatrix} 0.83 & 0.83 \\ 0.17 & 0.17 \end{bmatrix}
\]
Thus, the normalized relative weights are \( w_1 = (0.83 + 0.83)/2 = 0.83 \) and \( w_2 = (0.17 + 0.17)/2 = 0.17 \). Matrix \( \tilde{A} \) is consistent (since \( \tilde{A} \) is a \((2 \times 2)\) matrix). The weighting problem is therefore formulated as:

\[
P(\tilde{w}) = \max(w_1 z_1 + w_2 z_2) = \max 0.83(-x_1^2 + 6x_1x_2 + x_2 + x_1x_4x_5) + 0.17(x_2^2 + x_3x_4x_5)
\]

subject to

\[
x_1 - x_2 + x_3 + x_4 - x_5 \leq 40; \quad (x_3 - 3)^2 + (x_4 - 2)^2 = 13; \quad (x_1 - 4)^2 + x_2^2 \leq 16;
\]

\[
(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 - (x_4 - x_5)^2 \geq 25;
\]

\[
x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0.
\]

5 \leq x_1 \leq 25, \quad 2 \leq x_2 \leq 9.5, \quad 0 \leq x_3 \leq 9, \quad 0 \leq x_4 \leq 7.

The non-dominated solution set is generated by parametrically varying the weights and is tabulated below:

<table>
<thead>
<tr>
<th>( w_1, w_2 )</th>
<th>( x_1, x_2, x_3, x_4, x_5 )</th>
<th>( P(w) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>8.0,5.35,4.74,12.98</td>
<td>328.8</td>
</tr>
<tr>
<td>0.1,0.9</td>
<td>8.0,5.21,4.85,12.99</td>
<td>352.13</td>
</tr>
<tr>
<td>0.5,0.5</td>
<td>8.0,4.5,5.32,13.02</td>
<td>461.58</td>
</tr>
<tr>
<td>0.06,0.4</td>
<td>8.0,4.13,5.42,13.02</td>
<td>494.11</td>
</tr>
<tr>
<td>0.7,0.3</td>
<td>8.0,3.85,5.5,13.02</td>
<td>529.04</td>
</tr>
</tbody>
</table>

Example-2:

In this section we present numerical example to demonstrate the solution procedures by proposed approach to solve bi-level quadratic fractional programming problem (BLQFPP). The following example considered by Mishra [20] is again used to demonstrate the solution procedures and clarify the effectiveness of the proposed approach:

Consider the following BLQFPP:

\[
\begin{align*}
\text{Max } z_1 &= \frac{10x_1^2 + 15x_2^2 + 5}{x_1^2 + 2x_2^2 + 1} \\
\text{Max } z_2 &= \frac{25x_1^2 + 9x_2^2}{2x_1^2 + x_2^2 + 1}
\end{align*}
\]

subject to

\[
\begin{align*}
4x_1 - 5x_2 \leq 15; \\
3x_1 - x_2 \leq 21; \\
2x_1 + x_2 \leq 27; \\
x_1 + 4x_2 \leq 45; \\
x_1 + 3x_2 \leq 30; \\
x_1 \geq 0, x_2 \geq 0.
\end{align*}
\]

Solution by proposed approach:

\( z_1 = 9.6, \quad z_2 = 12.20, \quad x_1 = 5.34 \) and \( x_2 = 1.27 \)

VII. Conclusion

In this paper we consider the solution of a bi-level linear and non-linear programming problems by Weighting method. The Weighting method determines a subset of the complete set of non-dominated solutions. From this subset the DM chooses the most satisfying solution, making implicit trade-offs between objective functions based on some previously un-indicated or non-quantifiable criteria. Sometimes a feasible solution may not exist for a given combination of normalized weights. We also observe that even though we vary the weight vector, the solution remains more or less the same. Thus the non-dominated solution set reduces to a point (almost). The numerical result shows the proposed algorithm is feasible and efficient, can find global optimal solutions with less computational burden.

References


