Sum Square Difference Product Prime Labeling of Some Tree Graphs

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Abstract: Sum square difference product prime labeling of a graph is the labeling of the vertices with \{0,1,2,--------,p-1\} and the edges with absolute difference of the square of the sum of the labels of the incident vertices and product of the labels of the incident vertices. The greatest common incidence number of a vertex ( gc in) of degree greater than one is defined as the greatest common divisor of the labels of the incident edges. If the gc in of each vertex of degree greater than one is one, then the graph admits sum square difference product prime labeling. Here we identify some trees for sum square difference product prime labeling.

Keywords: Graph labeling, greatest common incidence number, sum square, trees, prime labeling.

I. Introduction

All graphs in this paper are trees. The symbol V(G) and E(G) denotes the vertex set and edge set of a graph G. The graph whose cardinality of the vertex set is called the order of G, denoted by p and the cardinality of the edge set is called the size of the graph G, denoted by q. A graph with p vertices and q edges is called a (p,q)-graph.

A graph labeling is an assignment of integers to the vertices or edges. Some basic notations and definitions are taken from [2],[3] and [4]. Some basic concepts are taken from [1] and [2]. In this paper we investigated sum square difference product prime labeling of some trees.

Definition 1.1 Let G be a graph with p vertices and q edges. The greatest common incidence number ( gc in) of a vertex of degree greater than or equal to 2, is the greatest common divisor (gcd) of the labels of the incident edges.

II. Main Results

Definition 2.1 Let G = (V(G),E(G)) be a graph with p vertices and q edges. Define a bijection 
\[ f : V(G) \rightarrow \{0,1,2,3,--------,p-1\} \] by 
\[ f(v_i) = i-1 \] , for every i from 1 to p and define a 1-1 mapping 
\[ f^*_sdppl : E(G) \rightarrow \text{set of natural numbers } N \] by 
\[ f^*_sdppl(\{u,v\}) = |(f(u) +f(v))^2 - f(u)f(v)|. \]

The induced function 
\[ f^*_sdppl \] is said to be sum square difference product prime labeling, if for each vertex of degree at least 2, the greatest common incidence number is 1.

Definition 2.2 A graph which admits sum square difference product prime labeling is called a sum square difference product prime graph.

Theorem 2.1 Comb graph \( P_n \times K_1 \) admits sum square difference product prime labeling.

Proof: Let G = \( P_n \times K_1 \) and let \( v_1,v_2,-------------,v_{2n} \) are the vertices of G

Here |V(G)| = 2n and |E(G)| =2n-1

Define a function 
\[ f : V \rightarrow \{0,1,2,3,--------,2n-1 \} \] by
\[ f(v_i) = i-1 , \quad i = 1,2,--------,2n \]

Clearly f is a bijection.

For the vertex labeling f, the induced edge labeling \( f^*_sdppl \) is defined as follows:
\[ f^*_sdppl(v_i v_{i+1}) = (2n-1)^2 - (2n-2)(i+1), \quad i = 1,2,--------,n \]

Clearly \( f^*_sdppl \) is an injection.

\( gc in \) of \( (v_i) \)

\[ gc in (v_i) = \text{gcd of } \{ f^*_sdppl(v_i v_{i+1}) , f^*_sdppl(v_{i+1} v_{i+2}) \} \]
\[ = \text{gcd of } \{ 3i^2 - 3i + 1, 3i^2 - 3i + 1 \} \]
\[ = \text{gcd of } \{ 3(2n-1)^2 - (2n-2)(i+1), 3(2n-2)^2 - (2n-3)(i+1) \} \]
\[ = \text{gcd of } \{ 3i, 3i \} \]
\[ = \text{gcd of } \{ 3i, 3i \} \]
\[ = \text{gcd of } \{ 3, 3 \} \]
\[ = 1, \quad i = 1,2,--------,n. \]

So, \( gc in \) of each vertex of degree greater than one is 1.

Hence \( P_n \times K_1 \) admits sum square difference product prime labeling.

\[\square\]
Theorem 2.2 Star $K_{1,n}$ admits sum square difference product prime labeling.
Proof: Let $G = K_{1,n}$ and let $u,v_1,v_2,\ldots,v_n$ are the vertices of $G$
Here $|V(G)| = n+1$ and $|E(G)| = n$
Define a function $f : V \rightarrow \{0,1,2,3,\ldots,n\}$ by

\[
    f(v_i) = i , \quad i = 1,2,\ldots,n
\]

$f(u) = 0$
Clearly $f$ is a bijection.
For the vertex labeling $f$, the induced edge labeling $f^*_{ssdppl}$ is defined as follows

\[
    f^*_{ssdppl}(u,v_i) = i^2 , \quad i = 1,2,\ldots,n
\]
Clearly $f^*_{ssdppl}$ is an injection.
$gcd$ of $(a)$
$= 1$.
So, $gcd$ of each vertex of degree greater than one is 1.
Hence $K_{1,n}$ admits sum square difference product prime labeling.  

Theorem 2.3 Bistar $B(m,n)$ admits sum square difference product prime labeling.
Proof: Let $G = B(m,n)$ and let $a,b,v_1,v_2,\ldots,v_{m+n}$ are the vertices of $G$
Here $|V(G)| = m+n$ and $|E(G)| = m+n+1$
Define a function $f : V \rightarrow \{0,1,2,3,\ldots,m+n\}$ by

\[
    f(v_i) = i+1 , \quad i = 1,2,\ldots,m
\]

\[
    f(u_i) = m+i+1 , \quad i = 1,2,\ldots,n
\]

\[
    f(a) = 0 , \quad f(b) = 1
\]
Clearly $f$ is a bijection.
For the vertex labeling $f$, the induced edge labeling $f^*_{ssdppl}$ is defined as follows

\[
    f^*_{ssdppl}(a,v_i) = (i+1)^2 , \quad i = 1,2,\ldots,m
\]
\[
    f^*_{ssdppl}(b,u_i) = (m+i+2)^2 - (m+i+1) , \quad i = 1,2,\ldots,n
\]
\[
    f^*_{ssdppl}(a,b) = 1
\]
Clearly $f^*_{ssdppl}$ is an injection.
$gcd$ of $(a)$
$= 1$.
$gcd$ of $(b)$
$= 1$.
So, $gcd$ of each vertex of degree greater than one is 1.
Hence $B(m,n)$ admits sum square difference product prime labeling.  

Theorem 2.4 Subdivision graph of Star $K_{1,n}$ admits sum square difference product prime labeling.
Proof: Let $G = Sd(K_{1,n})$ and let $a,b,v_1,v_2,\ldots,v_{2n}$ are the vertices of $G$
Here $|V(G)| = 2n+1$ and $|E(G)| = 2n$
Define a function $f : V \rightarrow \{0,1,2,3,\ldots,2n\}$ by

\[
    f(v_i) = i , \quad i = 1,2,\ldots,2n
\]

\[
    f(a) = 0
\]
Clearly $f$ is a bijection.
For the vertex labeling $f$, the induced edge labeling $f^*_{ssdppl}$ is defined as follows

\[
    f^*_{ssdppl}(a,v_{2i-1}) = (2i-1)^2 , \quad i = 1,2,\ldots,n
\]
\[
    f^*_{ssdppl}(a,v_{2i}) = 12i^2 - 6i + 1 , \quad i = 1,2,\ldots,n
\]
Clearly $f^*_{ssdppl}$ is an injection.
$gcd$ of $(a)$
$= 1$.
$gcd$ of $(v_{2i})$
$= gcd of $\{f^*_{ssdppl}(a,v_{2i-1}) , f^*_{ssdppl}(v_{2i-1},v_{2i})\}$
$= gcd of \{ (2i-1)^2 , 12i^2-6i+1\}$
$= gcd of (2i-1,6i(2i+1))$
$= 1$.
$= 1,2,\ldots,n$.
So, $gcd$ of each vertex of degree greater than one is 1.
Hence $Sd(K_{1,n})$ admits sum square difference product prime labeling.  

Theorem 2.5 Coconut tree graph $CT(m,n)$ admits sum square difference product prime labeling.
Proof: Let $G = CT(m,n)$ and let $v_1,v_2,\ldots,v_{mn}$ are the vertices of $G$
Here $|V(G)| = m+n$ and $|E(G)| = m+n+1$
Define a function $f : V \rightarrow \{0,1,2,3,\ldots,m+n\}$ by

\[
    f(v_i) = i-1 , \quad i = 1,2,\ldots,m+n
\]
Clearly $f$ is a bijection.
For the vertex labeling $f$, the induced edge labeling $f^*_{ssdppl}$ is defined as follows

\[
    f^*_{ssdppl}(v_i,v_{i+1}) = 3i^2 - 3i + 1 , \quad i = 1,2,\ldots,m-1
\]
\[
    f^*_{ssdppl}(v_m,v_{m+i}) = (2m+i-2)^2 - (m+i-1)(m-1) , \quad i = 1,2,\ldots,n
\]
Clearly $f^*_{ssdppl}$ is an injection.
Define a function \( f : V \) of each vertex of degree greater than one is 1.

Hence CT\((m,n)\), admits sum square difference product prime labeling.

**Theorem 2.6** Centipede graph \( C(2,n) \) admits sum square difference product prime labeling.

Proof: Let \( G = C(2,n) \) and let \( v_1, v_2, \ldots, v_{3n} \) are the vertices of \( G \).

Here \(|V(G)| = 3n\) and \(|E(G)| = 3n-1\).

Define a function \( f : V \rightarrow \{1,2,3,\ldots,3n\} \) by

\[ f(v_i) = i-1, \quad i = 1,2,\ldots,3n \]

Clearly \( f \) is a bijection.

For the vertex labeling \( f \), the induced edge labeling \( f^*_{\text{ssdppl}} \) is defined as follows

\[
\begin{align*}
    f^*_{\text{ssdppl}}(v_i, v_{i+1}) &= 3i^2 - 3i + 1, & i = 1,2,\ldots,n-1 \\
    f^*_{\text{ssdppl}}(v_i, v_{n+i}) &= (n+2i-1)^2 - (n+i)i, & i = 1,2,\ldots,n-2 \\
    f^*_{\text{ssdppl}}(v_i, v_{2n-i+1}) &= (3n^2 - 3n - 1)i, & i = 1,2,\ldots,n
\end{align*}
\]

Clearly \( f^*_{\text{ssdppl}} \) is an injection.

So, \( \text{gcin} \) of each vertex of degree greater than one is 1.

Hence \( C(2,n) \), admits sum square difference product prime labeling.

**Example 2.1**

\[ G = C(2,4) \]

\[ \text{fig 2.1} \]

**Theorem 2.7** Twig graph \( T_n(n) \) admits sum square difference product prime labeling.

Proof: Let \( G = T_n(n) \) and let \( v_1, v_2, \ldots, v_{3n} \) are the vertices of \( G \).

Here \(|V(G)| = 3n\) and \(|E(G)| = 3n-5\).

Define a function \( f : V \rightarrow \{0,1,2,3,\ldots,3n-5\} \) by

\[ f(v_i) = i-1, \quad i = 1,2,\ldots,3n-4 \]

Clearly \( f \) is a bijection.

For the vertex labeling \( f \), the induced edge labeling \( f^*_{\text{ssdppl}} \) is defined as follows

\[
\begin{align*}
    f^*_{\text{ssdppl}}(v_i, v_{i+1}) &= 3i^2 - 3i + 1, & i = 1,2,\ldots,n-1 \\
    f^*_{\text{ssdppl}}(v_i, v_{n+i}) &= (n+2i-1)^2 - (n+i)i, & i = 1,2,\ldots,n-2 \\
    f^*_{\text{ssdppl}}(v_i, v_{2n-i+1}) &= (2n-3+3i)i, & i = 1,2,\ldots,n-2
\end{align*}
\]

Clearly \( f^*_{\text{ssdppl}} \) is an injection.

So, \( \text{gcin} \) of each vertex of degree greater than one is 1.

Hence \( T_n(n) \), admits sum square difference product prime labeling.

**Theorem 2.8** H-graph of path \( P_n \) admits sum square difference product prime labeling.

Proof: Let \( G = H(P_n) \) and let \( v_1, v_2, \ldots, v_{2n} \) are the vertices of \( G \).

Here \(|V(G)| = 2n\) and \(|E(G)| = 2n-1\).

Define a function \( f : V \rightarrow \{0,1,2,3,\ldots,2n-1\} \) by

\[ f(v_i) = i-1, \quad i = 1,2,\ldots,2n \]

Clearly \( f \) is a bijection.

For the vertex labeling \( f \), the induced edge labeling \( f^*_{\text{ssdppl}} \) is defined as follows

\[
\begin{align*}
    f^*_{\text{ssdppl}}(v_i, v_{i+1}) &= 3i^2 - 3i + 1, & i = 1,2,\ldots,n-1 \\
    f^*_{\text{ssdppl}}(v_i, v_{n+i}) &= (2n+2i-1)^2 - (n+i)(n+i-1), & i = 1,2,\ldots,n-1
\end{align*}
\]

**Case (i) \( n \) is odd**

\[
\begin{align*}
    f^*_{\text{ssdppl}}(v_{\frac{n+1}{2}}, v_{\frac{3n+1}{2}}) &= (2n-1)^2 - \left( \frac{3(n-1)(n-1)}{4} \right)
\end{align*}
\]
Case(ii) \( n \) is even

\[
f_{ssdpl}(\frac{v_{(n+2)}}{T}, \frac{v_{(n)}}{T}) = (2n-1)^2 - \frac{(3n-2)(n)}{4}
\]

Clearly \( f_{ssdpl} \) is an injection.

- \( \text{gein} \) of \( (v_{i+1}) \):
  - \( i = 1,2,\ldots,n-2 \)

- \( \text{gein} \) of \( (v_{n+i+1}) \):
  - \( i = 1,2,\ldots,n-2 \)

So, \( \text{gein} \) of each vertex of degree greater than one is 1.

Hence \( H(P_n) \) admits sum square difference product prime labeling.  \( \blacksquare \)

**Theorem 2.9** Tensor product of star \( K_{1,n} \) and path \( P_2 \) admits sum square difference product prime labeling, when \( n > 2 \).

**Proof:** Let \( G = K_{1,n} \otimes P_2 \) and let \( a,b,v_1,v_2,\ldots,v_n,u_1,u_2,\ldots, u_n \) are the vertices of \( G \).

Here \( |V(G)| = 2n+2 \) and \( |E(G)| = 2n \).

Define a function \( f : V \rightarrow \{0,1,2,3,\ldots,2n+1\} \) by

- \( f(v_i) = i+1 \), \( i = 1,2,\ldots,n \)
- \( f(u_i) = n+i+1 \), \( i = 1,2,\ldots,n \)
- \( f(a) = 0 \), \( f(b) = 1 \).

Clearly \( f \) is a bijection.

For the vertex labeling \( f \), the induced edge labeling \( f_{ssdpl}^* \) is defined as follows

- \( f_{ssdpl}^*(a \ u_{i}) = (n+i+1)^2 \), \( i = 1,2,\ldots,n \)
- \( f_{ssdpl}^*(b \ v_{i}) = i^2+3i+3 \), \( i = 1,2,\ldots,n \)

Clearly \( f_{ssdpl}^* \) is an injection.

- \( \text{gein} \) of \( (a) \) = 1.
- \( \text{gein} \) of \( (b) \) = 1.

So, \( \text{gein} \) of each vertex of degree greater than one is 1.

Hence \( K_{1,n} \otimes P_2 \) admits sum square difference product prime labeling.  \( \blacksquare \)

### III. References


