Multiverse Cosmology from Exact Solution of Generalized Modified Wheeler-De Witt Equation

Anjan Kumar Chowdhury
Department of Physics
University of Chittagong
Chittagong-4331
BANGLADESH

Abstract: Minisuperspace Wheeler-DeWitt equation has been solved for various types of potentials. A simple potential has been used in [5,6] where it has been shown that the universe expands from zero volume reaching a maximum size and then it recollapses to zero volume again. In this paper we have used a general type of potential and found an exact solution of the generalized modified minisuperspace Wheeler-DeWitt equation. The solution indicates that there might be infinite types and numbers of universes which gives the taste of multiverse cosmology and those universes expand or contract exponentially. The solution also avoids the Big Bang singularity.

Keywords: multiverse cosmology; quantum gravity; Wheeler DeWitt equation; exact solution; singularity

I. Introduction

It is generally accepted that the wavefunction of a quantum particle contains all information regarding that particle. If the Big Bang theory is correct then the universe starts expansion from zero volume or literally from a point particle. The universe was congealed with immense energy at the time of its birth. So, there must be quantum fluctuations at that time and the laws of quantum mechanics should be applied to that super particle. From that point of view the wavefunction of the universe was conjectured [1-5]. When quantum mechanics and general theory of relativity are combined to a certain satisfactory level, like Schrödinger equation in quantum mechanics, one gets a second order differential functional equation, named as, Wheeler-DeWitt (WDW) equation, which is also the basic equation of quantum gravity. Hence, it is expected that all information regarding the universe can be extracted from the wavefunction of the universe. WDW equation is a functional differential equation for the wavefunction of the universe \( \Psi[h_{ij}] \), which is a functional of the three-geometries \( h_{ij} \), given by,

\[
\mathcal{H}\Psi[h_{ij}] = \left\{ -G_{ijkl} \frac{\partial^2}{\partial h_{ij} \partial h_{kl}} - \sqrt{h} \frac{3}{2} R \right\} \Psi[h_{ij}] = 0
\]  

(1)

where, \( G_{ijkl} \) is known as the De Witt metric or (5+1) dimensional metric on superspace with signature \((-+++++)\), given by

\[
G_{ijkl} = \frac{1}{2\sqrt{h}} (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl}).
\]

(2)

The solution of equation (1) is notoriously difficult. One way to get some primary information regarding the universe is to reduce the dimension of the equation (1). The most reduced and simplified equation is known as the “minisuperspace” WDW equation which has only two dimensions with homogeneous and isotropic manifolds. The corresponding minisuperspace WDW equation [7-10] can be given by

\[
\frac{1}{2} \left[ \frac{1}{a^2} \frac{\partial}{\partial a} \left( a^2 \frac{\partial}{\partial a} \right) - \frac{\partial^2}{\partial \chi^2} + U(a, \chi) - 2\epsilon_0 \right] \Psi(a, \chi) = 0,
\]

(3)

where, \( 2\epsilon_0 \) is an arbitrary constant for matter-energy renormalization and \( U(a, \chi) \) is the potential and is given by

\[
U(a, \chi) = \lambda \ a^4 - a^2 + \chi^2
\]

(4)

where, \( a \) and \( \chi \) be the scale factor and conformally invariant scalar field respectively. The potential described by (4) is a simple one. In this paper we generalize this potential for some arbitrary parameters and found an exact
solution of the generalized modified minisuperspace WDW equation. The solution of this equation implies that there are infinite numbers of universes of different types depending on the parameters considered.

II. Generalized modified Minisuperspace Wheeler-DeWitt Equation

Let us generalize the potential function (4) as

$$U(a, \chi) = \beta(\chi^2 - a^2) + \lambda(\chi^2 - a^2)^n' + \sigma(\chi^2 - a^2)^n$$  \hspace{1cm} (5)

where $\beta$, $\lambda$ and $\sigma$ are arbitrary parameters and $n$ and $n'$ are integers with some unique relations among them whose will be shown later (Fig. 1). $\lambda$ is related to Hubble constant $H$ by $\lambda = H^2$. Hence the modified Wheeler-DeWitt equation in minisuperspace can be given by:

$$\frac{\partial^2 \Psi}{\partial a^2} + \frac{p}{a} \frac{\partial \Psi}{\partial a} + \beta(\chi^2 - a^2)\Psi + \lambda(\chi^2 - a^2)^n' \Psi - \frac{\partial^2 \Psi}{\partial \chi^2} + \sigma(\chi^2 - a^2)^n \Psi - (2\epsilon_0 \pm E)\Psi = 0.$$  \hspace{1cm} (6)

Figure 1: Graphical representation of general WDW potential (Eq. 5) for $n' = 4$.

For convenience, let us write $\partial \Psi/\partial a \equiv \Psi_a$, $\partial^2 \Psi/\partial a^2 \equiv \Psi_{aa}$ and $\partial^2 \Psi/\partial \chi^2 \equiv \Psi_{\chi\chi}$ and equation (6) becomes

$$\{\psi_{aa} + \frac{p}{a} \psi_a + \beta(\chi^2 - a^2)\Psi + \lambda(\chi^2 - a^2)^n' \Psi - \psi_{\chi\chi} + \sigma(\chi^2 - a^2)^n \Psi - (2\epsilon_0 \pm E)\Psi\} = 0$$  \hspace{1cm} (7)

Equation (7) is the modified minisuperspace WDW equation to be solved exactly.

III. Solution of Generalized modified Minisuperspace Wheeler-DeWitt Equation

Let us consider a general trial solution of equation (7)

$$\Psi(a, \chi) = \{q_0 + \beta(\chi^2 - a^2)^s\} \exp\{\alpha(\chi^2 - a^2)^m\}.$$  \hspace{1cm} (8)

where $q_0$ be a constant, $\alpha$ be an arbitrary parameter and $s$ & $m$ are any integers. Differentiating with respect to $a$ we get

$$\Psi_a = \{-2s\beta a(\chi^2 - a^2)^{s-1} - 2maa(\chi^2 - a^2)^{m-1}[q_0 + \beta(\chi^2 - a^2)^s]\} \exp\{\alpha(\chi^2 - a^2)^m\}$$  \hspace{1cm} (9)
Similarly,
\[
\Psi_{aa} = (-2s\beta (x^2 - a^2)^{s-1} + 4s(s-1)\beta a^2 (x^2 - a^2)^{s-2} - 2ma^2 (x^2 - a^2)^{m-1} [q_0 + \beta (x^2 - a^2)^s] 
+ 4m(m-1)\alpha a^2 (x^2 - a^2)^{m-2} [q_0 + \beta (x^2 - a^2)^s] + 4ma\beta sa^2 (x^2 - a^2)^{m+s-2} 
+ [4msa\beta a^2 (x^2 - a^2)^{m+s-2} + 4m^2a^2a^2 (x^2 - a^2)^{2m-2} 
\times [q_0 + \beta (x^2 - a^2)^s])] \exp[\alpha (x^2 - a^2)^m]
\]  
(10)

Differentiating (8) with respect to \( x \) we get
\[
\Psi_x = (2s\beta x (x^2 - a^2)^{s-1} + 2ma \beta (x^2 - a^2)^{m-1} [q_0 + \beta (x^2 - a^2)^s]) \exp[\alpha (x^2 - a^2)^m]
\]  
(11)

Similarly,
\[
\Psi_{xx} = (2s\beta (x^2 - a^2)^{s-1} + 4s(s-1)\beta x^2 (x^2 - a^2)^{s-2} 
+ [2ma \beta (x^2 - a^2)^{m-1} + 4m(m-1)\alpha x^2 (x^2 - a^2)^{m-2} [q_0 + \beta (x^2 - a^2)^s] 
+ 4msa\beta \beta x^2 (x^2 - a^2)^{m+s-2} + 2s\beta x (x^2 - a^2)^{s-1} + 2ma \beta (x^2 - a^2)^{m-1} [q_0 + \beta (x^2 - a^2)^s] 
\times 2ma \beta (x^2 - a^2)^{m-1} \exp[\alpha (x^2 - a^2)^m]
\]  
(12)

Putting the values of \( \Psi, \Psi_a, \Psi_{aa} \) and \( \Psi_{xx} \) into equation (7) and simplifying after canceling the exponential term we get
\[
\{-4s\beta (x^2 - a^2)^{s-1} - 4s(s-1)\beta (x^2 - a^2)^{s-1} - 2ma \beta (x^2 - a^2)^{m-1} [q_0 + \beta (x^2 - a^2)^s] 
- 4m(m-1)\alpha (x^2 - a^2)^{m-1} [q_0 + \beta (x^2 - a^2)^s] 
- 8msa\beta (x^2 - a^2)^{m+s-1} - 4m^2a^2 (x^2 - a^2)^{2m-1} [q_0 + \beta (x^2 - a^2)^s] 
- 2ma (x^2 - a^2)^{m-1} [q_0 + \beta (x^2 - a^2)^s] 
- p [2s\beta (x^2 - a^2)^{s-1} + 2ma (x^2 - a^2)^{m-1} [q_0 + \beta (x^2 - a^2)^s]]
\]  
(13)

Let us consider \( m = s, \ n = 2m - 1, \ n = 2n' + 1 \) and \( n' = m - 1 = s - 1 \). Now equating the powers we get
\[
(\chi^2 - a^2)^{2m+s-1}: [-4m^2a^2\beta + \alpha \beta] = 0
\]  
(14)

\[
(\chi^2 - a^2)^{m+s-1}: [-4m(m-1)\beta - 8msa\beta - 2pm\alpha \beta + \lambda \beta] = 0
\]  
(15)

\[
(\chi^2 - a^2)^{n'}: [-4s\beta - 4s(s-1)\beta - 4maq_0 - 4m(m-1)\alpha q_0 - 2ps\beta - 2mpaq_0 + \lambda q_0] = 0
\]  
(16)

Equations (14-16) give:
\[
\sigma = 4m^2a^2,
\]  
(17)

\[
\lambda = 2s(4s + p)\alpha,
\]  
(18)

\[
(2s + p)\beta = 2s\alpha q_0,
\]  
(19)
Under these conditions the solution (8) satisfies equation (7). Writing \((\chi^2 - a^2) = r\) we get from (8)
\[
\Psi(r) = \{q_0 + \beta r^s\} \exp(\alpha r^s).
\] (20)

**Figure 2:** The graphical representation of the solution (8) for \(s = 4\) when \(\alpha\) and \(\beta\) take +ve values.

**Figure 3:** The graphical representation of the solution (8) for \(s = 4\) when \(\alpha\) is +ve and \(\beta\) is -ve.

**Figure 4:** The graphical representation of the solution (20) for \(s = 1 \ldots 10\) when \(\alpha\) and \(\beta\) take +ve values.
IV. Discussions

The general solution given by (20) for \( s = 0, 1, 2, 3 \ldots \), with individual properties depending upon the parameters considered in the equations (17-19). This can be visualized as multiverse scenario. The expansion and contraction both are exponential according to equation to (20). Every integer corresponding to \( s \) can be regarded as quantum numbers of the individual universes. Very interesting point comes when one considers \( s = 0 \). This shows that the universe starts expansion from a finite volume. It is generally believed that the universe starts expansion from the Planck size (\( \sim 10^{-33} \text{ cm} \)). Hence it is observed that this solution avoids the Big Bang singularity. When the universe remained at its ground state the question might peep that why did the universe start expansion? This point is not so clear; but one can assume that at the ground state the universe possessed highest possible degrees of symmetries which were broken by the violent quantum fluctuation due to Heisenberg’s uncertainty principle. The quantum fluctuation might initiate the Big Bang and the expansion of the universe. The solution (20) does not predict the cause of the Big Bang or expansion, but tells the universes how to expand. Another important aspect of our exact solution is that the solution is independent of the energy considered. Hence it is meaningless to ask the question that from where the universe comes or what happened before Big Bang.

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References: